DECORATIONS ON GEOMETRIC CRYSTALS AND MONOMIAL REALIZATIONS OF CRYSTAL BASES FOR CLASSICAL GROUPS

TOSHIKI NAKASHIMA

Abstract. We shall describe explicitly the decoration functions for certain decorated geometric crystals of classical groups and we shall show that they are represented in terms of monomial realizations of crystal bases.

Contents

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1. Introduction

Since the theory of crystal bases was invented by Kashiwara([6, 7]), there have been several kinds of realizations for crystal bases, *e.g.,* tableaux, paths, polytopes, monomials, etc. In the article, the monomial realization of crystal bases, which is introduced by Nakajima [13] and refined by Kashiwara [9], will be treated and used to describe the decoration functions for the decorated geometric crystals ([2], see also below). Let $\mathcal{Y}$ be the set of Laurant monomials in doubly-indexed variables $\{Y_{m,i}\}$ $(m\in\mathbb{Z},\ i\in I:=\{1,2,\cdots,n\})$ as follows:

$$\mathcal{Y}:=\{Y=\prod_{m\in\mathbb{Z},i\in I}Y_{m,i}^{l_{m,i}}|l_{m,i}\in\mathbb{Z}\setminus\{0\}\text{ except for finitely many }(m,i)\}.$$

We can define the crystal structure on $\mathcal{Y}$ by the way as in 5.1 and it is shown that certain connected component $B(Y)\subset\mathcal{Y}$ including the highest monomial Y is isomorphic to the crystal $B(\lambda)$ where $\lambda=\mathrm{wt}(Y)$ is a dominant integral weight. For example, for type A_n and any integer m we have

$$B(\Lambda_1)\cong\{Y_{m,1},\frac{Y_{m,2}}{Y_{m+1,1}},\frac{Y_{m,3}}{Y_{m+1,2}},\cdots,\frac{Y_{m,n}}{Y_{m+1,n-1}},\frac{1}{Y_{m,n}}\},$$

where Λ_1 is the first fundamental weight and $Y_{m,1}$ is the highest monomial.

A geometric crystal is a sort of geometric lifting of Kashiwara's crystal bases ([1]), which is generalized to the affine/Kac-Moody settings ([11, 12, 15]). In this paper we do not treat such general settings and then we shall consider the simple classical settings below. Let $\mathfrak{g}$ be a simple complex Lie algebra, G the corresponding complex algebraic group, $B^\pm\subset G$ Borel subgroups and $U^\pm$ maximal unipotent subgroups such that $U^\pm\subset B^\pm$. The notion of decorated geometric crystals has been initiated by Berenstein and Kazhdan([2]). Let I be the index set of the simple roots. Associated with the Cartan matrix $A=(a_{i,j})_{i,j\in I}$, the decorated geometric crystal $\mathcal{X}=(\chi,f)$ is defined as a pair of geometric crystal $\chi=(X,\{e_i\}_i,\{\gamma_i\}_i,\{\varepsilon_i\}_i)$ and a certain special rational function f on the algebraic variety X satisfying the condition

$$f(e_i^c(x))=f(x)+(c-1)\varphi_i(x)+(c^{-1}-1)\varepsilon_i(x),$$

for any $i\in I$, where e_i^c is the unital rational $\mathbb{C}^\times$ action on X, and $\varphi_i:=\varepsilon_i\cdot\gamma_i$ is the rational functions on X. The function f is called the decoration (function) of the decorated geometric crystal $\mathcal{X}$. In [2] the ultra-discretization of the decoration is used to describe the Kashiwara's crystal base $B(\lambda)$, which is quite similar to the polyhedral realizations of crystal bases([14, 19]). This similarity let us conceive some link between crystal bases and the decorations. Indeed, in [18] we made this link clear for type A_n. Here we shall consider another link between them, which is the main purpose of this article. The purpose here is to present the explicit form of the decorations for the decorated geometric

crystals on $\mathbb{B}_{w_0}$ (see 4.4) and describe the decorations in terms of the monomial realizations of crystal bases([9, 13]). In [18], we presented the conjecture (see also Conjecture 6.8 below) and gave the positive answer for type A_n. To be more precise, we shall introduce certain part of the results in [18]. First, we consider the geometric crystal structure on the variety $\mathbb{B}_{w_0} := TB^-_{w_0}$ where $T \subset G$ is the maximal torus, w_0 is the longest element of the Weyl group and $B^-_{w_0} := B^- \cap U\overline{w}_0U$. Let $\chi_i : U \to \mathbb{C}$ be the elementary character and $\eta : G \to G$ the positive inverse (see 4.2). Then the decoration f_B on $\mathbb{B}_{w_0}$ is defined by the formula

$$f_B(g) := \sum_i \chi_i(\pi^+(w_0^{-1}g)) + \chi_i(\pi^+(w_0^{-1}\eta(g))),$$

where $\pi^+ : B^-U \to U$ is the projection. By the definition of f_B, it suffices to get the explicit form of $\chi_i(\pi^+(w_0^{-1}g))$ and $\chi_i(\pi^+(w_0^{-1}\eta(g)))$ for our purpose. Furthermore, the elementary characters are expressed by the "generalized minors $\Delta_{\gamma,\delta}$" ([3, 4, 5]) and then as in (6.2) we have

$$f_B(g) = \sum_i \frac{\Delta_{w_0\Lambda_i, s_i\Lambda_i}(g) + \Delta_{w_0s_i\Lambda_i,\Lambda_i}(g)}{\Delta_{w_0\Lambda_i,\Lambda_i}(g)},$$

where Λ_i is the i-th fundamental weight. We shall see the explicit forms of $\Delta_{w_0\Lambda_i,s_i\Lambda_i}(g)$ and $\Delta_{w_0s_i\Lambda_i,\Lambda_i}(g)$ as in (6.8) and (6.9). In most cases except for the spin representations of type B_n and D_n, it is performed by direct calculations. For the cases of the spin representations, we prepare the "triangles", which has some interesting combinatorial properties and is useful to calculate the above generalized minors. Then, we can find their relations to the monomial realizations of crystals (Sect.5).

In [18], we also describe the relations to the polyhedral realizations explicitly for type A_n though we do not treat that part herein. However, we strongly believe that there exist the relations similar to the ones for type A_n. As for the relations to the polyhedral realizations for other classical cases, we shall discuss in forthcoming papers.

The organization of the article is as follows: After the introduction in this section and the preliminaries in Sect.2, we review the explicit descriptions for the fundamental representation of the classical Lie algebras in Sect.3. In Sect.4, first we introduce the theory of decorated geometric crystals following [2]. Next, we define the decoration by using the elementary characters and certain special positive decorated geometric crystal on $\mathbb{B}_w = TB^-_w$. Finally, the ultra-discretization of TB^-_w is described explicitly. In Sect.5, the theory of monomial realizations would be introduced and we shall see some duality on monomial realizations. In Sect.6, we review the generalized minors and their relations to our elementary characters and certain bilinear forms. The main conjecture will be presented at the end of the section. In the last three sections, we describe the explicit form of decorations and express them in terms of the monomial realization of crystal bases, which means that the conjecture is positively resolved for the classical groups.

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2. Preliminaries and Notations

We list the notations used in this paper. Let $A = (\mathbf{a}_{ij})_{i,j\in I}$ be an indecomposable Cartan matrix with a finite index set I (though we can consider more general Kac-Moody setting.). Let $(\mathfrak{t}, \{\alpha_i\}_{i\in I}, \{h_i\}_{i\in I})$ be the associated root data satisfying $\alpha_j(h_i) = \mathbf{a}_{ij}$ where $\alpha_i \in \mathfrak{t}^*$ is a simple root and $h_i \in \mathfrak{t}$ is a simple coroot. Let $\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{t}, e_i, f_i (i \in I)\rangle$ be the simple Lie algebra associated with A over $\mathbb{C}$ and $\Delta = \Delta_+ \sqcup \Delta_-$ be the root system associated with $\mathfrak{g}$, where $\Delta_\pm$ is the set of positive/negative roots. Let $P \subset \mathfrak{t}^*$ be the weight lattice, $\langle h, \lambda\rangle = \lambda(h)$ the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$, and (α, β) be an inner product on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ and $\langle h_i, \lambda\rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$. Let $P^* = \{h \in \mathfrak{t} : \langle h, P\rangle \subset \mathbb{Z}\}$ and $P_+ := \{\lambda \in P : \langle h_i, \lambda\rangle \in \mathbb{Z}_{\geq 0}\}$. We call an element in P_+ a *dominant*

integral weight. Let $\{\Lambda_i | i \in I\}$ be the set of the fundamental weights satisfying $\langle h_i, \Lambda_j\rangle = \delta_{i,j}$ is a $\mathbb{Z}$-basis of P.

The quantum algebra $U_q(\mathfrak{g})$ is an associative $\mathbb{Q}(q)$-algebra generated by the e_i, f_i $(i \in I)$, and q^h $(h \in P^*)$ satisfying the usual relations, where we use the same notations for the generators e_i and f_i as the ones for $\mathfrak{g}$. The algebra $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i $(i \in I)$.

For the irreducible highest weight module of $U_q(\mathfrak{g})$ with the highest weight $\lambda \in P_+$, we denote it by $V(\lambda)$ and we denote its *crystal base* by $(L(\lambda), B(\lambda))$. Similarly, for the crystal base of the algebra $U_q^-(\mathfrak{g})$ we denote $(L(\infty), B(\infty))$ (see [6, 7]). Let $\pi_\lambda : U_q^-(\mathfrak{g}) \longrightarrow V(\lambda) \cong U_q^-(\mathfrak{g})/\sum_i U_q^-(\mathfrak{g}) f_i^{1+\langle h_i,\lambda\rangle}$ be the canonical projection and $\widehat{\pi}_\lambda : L(\infty)/qL(\infty) \longrightarrow L(\lambda)/qL(\lambda)$ be the induced map from π_λ. Here note that $\widehat{\pi}_\lambda(B(\infty)) = B(\lambda) \sqcup \{0\}$.

By the terminology *crystal* we mean some combinatorial object obtained by abstracting the properties of crystal bases. Indeed, crystal constitutes a set B and the maps $wt : B \longrightarrow P$, $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \longrightarrow B \sqcup \{0\}$ $(i \in I)$ satisfying several axioms (see [8],[19],[14]). In fact, $B(\infty)$ and $B(\lambda)$ are the typical examples of crystals.

3. Fundamental Representations

3.1. **Type A_n.** Let $V_1 := V(\Lambda_1)$ be the vector representation of $\mathfrak{sl}_{n+1}(\mathbb{C})$ with the standard basis $\{v_1, \cdots, v_{n+1}\}$, and $\{e_i, f_i, h_i\}_{i=1,\cdots,n}$ the Chevalley generators of $\mathfrak{sl}_{n+1}(\mathbb{C})$. Their actions on the basis vectors are as follows:

$$e_i v_j = \begin{cases} v_i & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \quad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \quad h_i v_j = \begin{cases} v_i & \text{if } j = i, \\ -v_{i+1} & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

3.2. **Type C_n.** Let $I := \{1, 2, \cdots, n\}$ be the index set of the simple roots of type C_n. The Cartan matrix $A = (\mathbf{a}_{i,j})_{i,j\in I}$ of type C_n is given by

$$\mathbf{a}_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1 \text{ and } (i,j) \neq (n-1, n) \\ -2 & \text{if } (i,j) = (n-1,n), \\ 0 & \text{otherwise.} \end{cases}$$

Here α_i $(i \neq n)$ is a short root and α_n is the long root. Let $\{h_i\}_{i\in I}$ be the set of the simple co-roots and $\{\Lambda_i\}_{i\in I}$ be the set of the fundamental weights satisfying $\alpha_j(h_i) = \mathbf{a}_{i,j}$ and $\Lambda_i(h_j) = \delta_{i,j}$.

First, let us describe the vector representation $V(\Lambda_1)$. Set $\mathbf{B}^{(n)} := \{v_i, v_{\overline{i}} | i = 1, 2, \cdots, n.\}$ and define $V(\Lambda_1) := \bigoplus_{v\in\mathbf{B}^{(n)}} \mathbb{C}v$. The weight of v_i is as follows:

$$\mathrm{wt}(v_i) = \begin{cases} \Lambda_i - \Lambda_{i-1} & \text{if } i = 1, \cdots, n, \\ \Lambda_{i-1} - \Lambda_i & \text{if } i = \overline{1}, \cdots, \overline{n}, \end{cases}$$

where $\Lambda_0 = 0$. The actions of e_i and f_i are given by:

$$f_i v_i = v_{i+1}, \quad f_i v_{\overline{i+1}} = v_{\overline{i}}, \quad e_i v_{i+1} = v_i, \quad e_i v_{\overline{i}} = v_{\overline{i+1}} \quad (1 \le i < n), \tag{3.2}$$

$$f_n v_n = v_{\overline{n}}, \qquad e_n v_{\overline{n}} = v_n, \tag{3.3}$$

and the other actions are trivial.

Let Λ_i be the i-th fundamental weight of type C_n. As is well-known that the fundamental representation $V(\Lambda_i)$ $(1 \le i \le n)$ is embedded in $V(\Lambda_1)^{\otimes i}$ with multiplicity free. The explicit form of the

highest(resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) of $V(\Lambda_i)$ is realized in $V(\Lambda_1)^{\otimes i}$ as follows:

$$\begin{aligned} u_{\Lambda_i} &= \sum_{\sigma\in\mathfrak{S}_i} \mathrm{sgn}(\sigma) v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(i)}, \\ v_{\Lambda_i} &= \sum_{\sigma\in\mathfrak{S}_i} \mathrm{sgn}(\sigma) v_{\overline{\sigma(i)}}\otimes\cdots\otimes v_{\overline{\sigma(1)}}, \end{aligned} \tag{3.4}$$

where $\mathfrak{S}_i$ is the i-th symmetric group.

We review the crystal $B(\Lambda_k)$ following [10, 16]. Set the order on the set $J=\{i,\overline{i}|1\leq i\leq n\}$ by

$$1<2<\cdots<n-1<n<\overline{n}<\overline{n-1}<\cdots<\overline{2}<\overline{1}$$

Then, the crystal $B(\Lambda_k)$ is described:

$$B(\Lambda_k)=\left\{[j_1,\cdots,j_k]\;\middle|\; \begin{array}{l} 1\leq j_1<\cdots<j_k\leq\overline{1}, \\ \text{if } \overline{j_a}=j_b=\overline{i},\ \text{then } a+b\leq i \end{array}\right\} \tag{3.5}$$

Note that in [10] the vector $[j_1,\cdots,j_k]$ is represented as the column Young tableau. Note also that the highest(resp. lowest) weight vector in $B(\Lambda_k)$ is $[1,2,\cdots,k]$ (resp. $[\overline{k},\overline{k-1},\cdots,\overline{2},\overline{1}]$).

3.3. **Type B_n.** Let $I:=\{1,2,\cdots,n\}$ be the index set of the simple roots of type B_n. The Cartan matrix $A=(\mathbf{a}_{i,j})_{i,j\in I}$ of type B_n is given by

$$\mathbf{a}_{i,j}=\begin{cases} 2 & \text{if } i=j, \\ -1 & \text{if } |i-j|=1 \text{ and } (i,j)\neq(n,n-1) \\ -2 & \text{if } (i,j)=(n,n-1), \\ 0 & \text{otherwise.} \end{cases}$$

Here α_i $(i\neq n)$ is a long root and α_n is the short root. Let $\{h_i\}_{i\in I}$ be the set of the simple co-roots and $\{\Lambda_i\}_{i\in I}$ be the set of the fundamental weights satisfying $\alpha_j(h_i)=\mathbf{a}_{i,j}$ and $\Lambda_i(h_j)=\delta_{i,j}$.

First, let us describe the vector representation $V(\Lambda_1)$ for B_n. Set $\mathbf{B}^{(n)}:=\{v_i,\,v_{\overline{i}}|i=1,2,\cdots,n\}\cup\{v_0\}$ and $V(\Lambda_1):=\bigoplus_{v\in\mathbf{B}^{(n)}}\mathbb{C}v$. The weight of v_i is as follows:

$$\begin{aligned} &\mathrm{wt}(v_i)=\Lambda_i-\Lambda_{i-1},\quad \mathrm{wt}(v_{\overline{i}})=\Lambda_{i-1}-\Lambda_i\quad (i=1,\cdots,n-1), \\ &\mathrm{wt}(v_n)=2\Lambda_n-\Lambda_{n-1},\quad \mathrm{wt}(v_{\overline{n}})=\Lambda_{n-1}-2\Lambda_n,\quad \mathrm{wt}(v_0)=0, \end{aligned}$$

where $\Lambda_0=0$. The actions of e_i and f_i are given by:

$$f_iv_i=v_{i+1},\quad f_iv_{\overline{i+1}}=v_{\overline{i}},\quad e_iv_{i+1}=v_i,\quad e_iv_{\overline{i}}=v_{\overline{i+1}}\quad (1\leq i<n), \tag{3.6}$$

$$f_nv_n=v_0,\quad f_nv_0=2v_{\overline{n}},\quad e_nv_0=2v_n,\quad e_nv_{\overline{n}}=v_0, \tag{3.7}$$

and the other actions are trivial. For $i=1,2,\cdots,n-1$, the i-th fundamental representation $V(\Lambda_i)$ is realized in $V(\Lambda_1)^{\otimes i}$ as the case C_n and their highest (resp. lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) is given by the formula (3.4).

The last fundamental representation $V(\Lambda_n)$ is called the "spin representation" whose dimension is 2^n. It is realized as follows: Set $V_{sp}^{(n)}:=\bigoplus_{\epsilon\in B_{sp}^{(n)}}\mathbb{C}\epsilon$ where

$$B_{sp}^{(n)}:=\{(\epsilon_1,\cdots,\epsilon_n)|\epsilon_i\in\{+,-\}\,(i=1,2,\cdots,n)\}.$$

Define the explicit actions of h_i, e_i and f_i on $V_{sp}^{(n)}$ by

$$h_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} \frac{\epsilon_i\cdot 1-\epsilon_{i+1}\cdot 1}{2}(\epsilon_1,\cdots,\epsilon_n), & \text{if } i\neq n, \\ \epsilon_n(\epsilon_1,\cdots,\epsilon_n) & \text{if } i=n, \end{cases} \tag{3.8}$$

$$f_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} (\cdots,\overset{i}{-},\overset{i+1}{+},\cdots) & \text{if } \epsilon_i=+,\ \epsilon_{i+1}=-,\ i\neq n, \\ (\cdots\cdots\cdots,\overset{n}{-}) & \text{if } \epsilon_n=+,\ i=n, \\ \quad 0 & \text{otherwise} \end{cases} \tag{3.9}$$

$$e_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} (\cdots,\overset{i}{+},\overset{i+1}{-},\cdots) & \text{if } \epsilon_i=-,\ \epsilon_{i+1}=+,\ i\neq n, \\ (\cdots\cdots\cdots,\overset{n}{+}) & \text{if } \epsilon_n=-,\ i=n, \\ \quad 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then the module $V_{sp}^{(n)}$ is isomorphic to $V(\Lambda_n)$ as a B_n-module.

Remark. We can associate the crystal structure on the set $B_{sp}^{(n)}$ by setting $\tilde{f}_i=f_i$ and $\tilde{e}_i=e_i$ in (3.9) and (3.10) respectively, which is also denoted by $B_{sp}^{(n)}$ and is isomorphic to $B(\Lambda_n)$.

3.4. **Type D_n.** Let $I:=\{1,2,\cdots,n\}$ be the index set of the simple roots of type D_n. The Cartan matrix $A=(\mathbf{a}_{i,j})_{i,j\in I}$ of type D_n is as follows:

$$\mathbf{a}_{i,j}=\begin{cases} 2 & \text{if } i=j, \\ -1 & \text{if } |i-j|=1 \text{ and } (i,j)\neq(n,n-1),\ (n-1,n),\ \text{or } (i,j)=(n-2,n),\ (n,n-2) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{h_i\}_{i\in I}$ be the set of the simple co-roots and $\{\Lambda_i\}_{i\in I}$ be the set of the fundamental weights satisfying $\alpha_j(h_i)=\mathbf{a}_{i,j}$ and $\Lambda_i(h_j)=\delta_{i,j}$.

First, let us describe the vector representation $V(\Lambda_1)$ for D_n. Set $\mathbf{B}^{(n)}:=\{v_i,v_{\overline{i}}|i=1,2,\cdots,n\}$. The weight of v_i is as follows:

$$\begin{aligned} &\mathrm{wt}(v_i)=\Lambda_i-\Lambda_{i-1},\quad \mathrm{wt}(v_{\overline{i}})=\Lambda_{i-1}-\Lambda_i\quad (i=1,\cdots,n-1), \\ &\mathrm{wt}(v_n)=\Lambda_{n-1}+\Lambda_n-\Lambda_{n-2},\quad \mathrm{wt}(v_{\overline{n}})=\Lambda_{n-2}-\Lambda_{n-1}+\Lambda_n, \end{aligned}$$

where $\Lambda_0=0$. The actions of e_i and f_i are given by:

$$f_iv_i=v_{i+1},\quad f_iv_{\overline{i+1}}=v_{\overline{i}},\quad e_iv_{i+1}=v_i,\quad e_iv_{\overline{i}}=v_{\overline{i+1}}\quad (1\leq i<n), \tag{3.11}$$

$$f_nv_n=v_{\overline{n-1}},\quad f_{n-1}v_{\overline{n}}=v_{\overline{n-1}},\quad e_{n-1}v_{\overline{n-1}}=v_{\overline{n}},\quad e_nv_{\overline{n-1}}=v_n, \tag{3.12}$$

and the other actions are trivial. For $i=1,2,\cdots,n-2$, the i-th fundamental representation $V(\Lambda_i)$ is realized in $V(\Lambda_1)^{\otimes i}$ as the cases B_n and C_n and their highest (resp.lowest) weight vector u_{Λ_i} (resp. v_{Λ_i}) is given by the formula (3.4).

The last two fundamental representations $V(\Lambda_{n-1})$ and $V(\Lambda_n)$ are also called the "spin representations" whose dimensions are 2^{n-1}. They are realized as follows: Set $V_{sp}^{(+,n)}$(resp. $V_{sp}^{(-,n)}$) $:=\bigoplus_{\epsilon\in B_{sp}^{(+,n)}(\text{resp. } B_{sp}^{(-,n)})}\mathbb{C}\epsilon$ where

$$B_{sp}^{(+,n)}(\text{resp. } B_{sp}^{(-,n)}):=\{(\epsilon_1,\cdots,\epsilon_n)|\epsilon_i\in\{+,-\},\epsilon_1\cdots\epsilon_n=+(\text{resp. } -)\}.$$

Define the explicit actions of h_i, e_i and f_i on $V_{sp}^{(\pm,n)}$ by

$$h_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} \frac{\epsilon_i\cdot 1-\epsilon_{i+1}\cdot 1}{2}(\epsilon_1,\cdots,\epsilon_n), & \text{if } i\neq n, \\ \frac{\epsilon_{n-1}\cdot 1+\epsilon_n\cdot 1}{2}(\epsilon_1,\cdots,\epsilon_n) & \text{if } i=n, \end{cases} \tag{3.13}$$

$$f_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} (\cdots,\overset{i}{-},\overset{i+1}{+},\cdots) & \text{if } \epsilon_i=+,\ \epsilon_{i+1}=-,\ i\neq n, \\ (\cdots\cdots\cdots,\overset{n-1}{-},\overset{n}{-}) & \text{if } \epsilon_{n-1}=+,\epsilon_n=+,\ i=n, \\ 0 & \text{otherwise} \end{cases} \tag{3.14}$$

$$e_i(\epsilon_1,\cdots,\epsilon_n) = \begin{cases} (\cdots,\overset{i}{+},\overset{i+1}{-},\cdots) & \text{if } \epsilon_i=-,\ \epsilon_{i+1}=+,\ i\neq n, \\ (\cdots\cdots\cdots,\overset{n-1}{+},\overset{n}{+}) & \text{if } \epsilon_{n-1}=-,\epsilon_n=-,\ i=n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Then the module $V_{sp}^{(+,n)}$ (resp. $V_{sp}^{(-,n)}$) is isomorphic to $V(\Lambda_n)$ (resp. $V(\Lambda_{n-1})$) as a D_n-module.

Remark. Similar to the case B_n, in this case we can associate the crystal structure on the set $B_{sp}^{(+,n)}$ (resp. $B_{sp}^{(-,n)}$) by setting $\tilde{f}_i=f_i$ and $\tilde{e}_i=e_i$ in (3.14) and (3.15) respectively, which is also denoted by $B_{sp}^{(\pm,n)}$ and is isomorphic to $B(\Lambda_n)$ (resp. $B(\Lambda_{n-1})$).

4. Decorated geometric crystals

The basic reference for this section is [1, 2, 15].

4.1. **Definitions.** Let $A=(\mathbf{a}_{ij})_{i,j\in I}$ be an indecomposable Cartan matrix. Let $\mathfrak{g}=\mathfrak{g}(A)=\langle\mathfrak{t},e_i,f_i(i\in I)\rangle$ be the simple Lie algebra associated with A over $\mathbb{C}$ as above and $\Delta=\Delta_+\sqcup\Delta_-$ be the root system associated with $\mathfrak{g}$. Define the simple reflections $s_i\in\mathrm{Aut}(\mathfrak{t})$ $(i\in I)$ by $s_i(h):=h-\alpha_i(h)h_i$, which generate the Weyl group W. Let G be the simply connected simple algebraic group over $\mathbb{C}$ whose Lie algebra is $\mathfrak{g}=\mathfrak{n}_+\oplus\mathfrak{t}\oplus\mathfrak{n}_-$, which is the usual triangular decomposition. Let $U_\alpha:=\exp\mathfrak{g}_\alpha$ $(\alpha\in\Delta)$ be the one-parameter subgroup of G. The group $U^\pm$ are generated by $\{U_\alpha|\alpha\in\Delta_\pm\}$. Here $U^\pm$ is a unipotent radical of G and $\mathrm{Lie}(U^\pm)=\mathfrak{n}_\pm$. For any $i\in I$, there exists a unique group homomorphism $\phi_i\colon SL_2(\mathbb{C})\to G$ such that

$$\phi_i\left(\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)=\exp(te_i),\ \phi_i\left(\begin{pmatrix}1 & 0\\ t & 1\end{pmatrix}\right)=\exp(tf_i) \qquad (t\in\mathbb{C}).$$

Set $\alpha_i^\vee(c):=\phi_i\left(\left(\begin{smallmatrix}c & 0\\ 0 & c^{-1}\end{smallmatrix}\right)\right)$, $x_i(t):=\exp(te_i)$, $y_i(t):=\exp(tf_i)$, $G_i:=\phi_i(SL_2(\mathbb{C}))$, $T_i:=\alpha_i^\vee(\mathbb{C}^\times)$ and $N_i:=N_{G_i}(T_i)$. Let T be a maximal torus of G which has P as its weight lattice and $\mathrm{Lie}(T)=\mathfrak{t}$. Let $B^\pm(\supset T,U^\pm)$ be the Borel subgroup of G. We have the isomorphism $\phi:W\xrightarrow{\sim}N/T$ defined by $\phi(s_i)=N_iT/T$. An element $\overline{s}_i:=x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i\in W=N_G(T)/T$.

Definition 4.1. Let X be an affine algebraic variety over $\mathbb{C}$, γ_i, ε_i, f $(i\in I)$ rational functions on X, and $e_i:\mathbb{C}^\times\times X\to X$ a unital rational $\mathbb{C}^\times$-action $(i\in I)$. A 5-tuple $\chi=(X,\{e_i\}_{i\in I},\{\gamma_i,\}_{i\in I},\{\varepsilon_i\}_{i\in I},f)$ is a G (or $\mathfrak{g}$)-*decorated geometric crystal* if

(i) $(\{1\}\times X)\cap dom(e_i)$ is open dense in $\{1\}\times X$ for any $i\in I$, where $dom(e_i)$ is the domain of definition of $e_i\colon\mathbb{C}^\times\times X\to X$.

(ii) The rational functions $\{\gamma_i\}_{i\in I}$ satisfy $\gamma_j(e_i^c(x))=c^{\mathbf{a}_{ij}}\gamma_j(x)$ for any $i,j\in I$.

(iii) The function f satisfies

$$f(e_i^c(x))=f(x)+(c-1)\varphi_i(x)+(c^{-1}-1)\varepsilon_i(x), \tag{4.1}$$

for any $i\in I$ and $x\in X$, where $\varphi_i:=\varepsilon_i\cdot\gamma_i$.

(iv) e_i and e_j satisfy the following relations:

$$
\begin{array}{ll}
e_i^{c_1}e_j^{c_2}=e_j^{c_2}e_i^{c_1} & \text{if } \mathbf{a}_{ij}=\mathbf{a}_{ji}=0,\\
e_i^{c_1}e_j^{c_1c_2}e_i^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1} & \text{if } \mathbf{a}_{ij}=\mathbf{a}_{ji}=-1,\\
e_i^{c_1}e_j^{c_1^2c_2}e_i^{c_1c_2}e_j^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^2c_2}e_i^{c_1} & \text{if } \mathbf{a}_{ij}=-2,\ \mathbf{a}_{ji}=-1,\\
e_i^{c_1}e_j^{c_1^3c_2}e_i^{c_1^2c_2}e_j^{c_1^3c_2^2}e_i^{c_1c_2}e_j^{c_2}=e_j^{c_2}e_i^{c_1c_2}e_j^{c_1^3c_2^2}e_i^{c_1^2c_2}e_j^{c_1^3c_2}e_i^{c_1} & \text{if } \mathbf{a}_{ij}=-3,\ \mathbf{a}_{ji}=-1.
\end{array}
$$

(v) The rational functions $\{\varepsilon_i\}_{i\in I}$ satisfy $\varepsilon_i(e_i^c(x))=c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x))=\varepsilon_i(x)$ if $\mathbf{a}_{i,j}=\mathbf{a}_{j,i}=0$.

We call the function f in (iii) the *decoration* of χ and the relations in (iv) are called *Verma relations*. If $\chi=(X,\{e_i\},\{\gamma_i\},\{\varepsilon_i\})$ satisfies the conditions (i), (ii), (iv) and (v), we call χ a *geometric crystal*.
Remark. The definitions of ε_i and φ_i are different from the ones in e.g., [2] since we adopt the definitions following [11, 12]. Indeed, if we flip $\varepsilon_i\to\varepsilon^{-1}$ and $\varphi_i\to\varphi^{-1}$, they coincide with ours.

4.2. **Characters.** Let $\widehat{U}:=\mathrm{Hom}(U,\mathbb{C})$ be the set of additive characters of U. The *elementary character* $\chi_i\in\widehat{U}$ and the *standard regular character* $\chi^{\mathrm{st}}\in\widehat{U}$ are defined as follows:

$$\chi_i(x_j(c))=\delta_{i,j}\cdot c\quad(c\in\mathbb{C},\ i\in I),\qquad \chi^{st}=\sum_{i\in I}\chi_i.$$

We also define the anti-automorphism $\eta:G\to G$ by

$$\eta(x_i(c))=x_i(c),\quad \eta(y_i(c))=y_i(c),\quad \eta(t)=t^{-1}\quad(c\in\mathbb{C},\ t\in T),$$

which is called the *positive inverse*([2]).

The rational function f_B on G is defined by

$$f_B(g)=\chi^{st}(\pi^+(w_0^{-1}g))+\chi^{st}(\pi^+(w_0^{-1}\eta(g))),\tag{4.2}$$

for $g\in B\overline{w}_0B$, where $\pi^+:B^-U\to U$ is the projection defined by $\pi^+(bu)=u$.

For a split algebraic torus T over $\mathbb{C}$, let us denote its lattice of (multiplicative)characters(resp. co-characters) by $X^*(T)$ (resp. $X_*(T)$). By the usual way, we identify $X^*(T)$ (resp. $X_*(T)$) with the weight lattice P (resp. the dual weight lattice P^*).

4.3. **Positive structure and ultra-discretization.** Let us review the notion positive structure and the ultra-discretization.

Definition 4.2. Let T,T' be split algebraic tori over $\mathbb{C}$.

(i) A regular function $f=\sum_{\mu\in X^*(T)}c_\mu\cdot\mu$ on T is *positive* if all coefficients c_μ are non-negative numbers. A rational function on T is said to be *positive* if there exist positive regular functions g,h such that $f=\frac{g}{h}$ $(h\neq0)$.
(ii) Let $f:T\to T'$ be a rational map between T and T'. Then we say that f is *positive* if for any $\xi\in X^*(T')$ we have that $\xi\circ f$ is positive in the above sense.

Note that if f,g are positive rational functions on T, then $f\cdot g$, f/g and $f+g$ are all positive.

Definition 4.3. Let $\chi=(X,\{e_i\}_{i\in I},\{\mathrm{wt}_i\}_{i\in I},\{\varepsilon_i\}_{i\in I},f)$ be a decorated geometric crystal, T' an algebraic torus and $\theta:T'\to X$ a birational map. The birational map θ is called *positive structure* on χ if it satisfies:

(i) For any $i\in I$ the rational functions $\gamma_i\circ\theta,\varepsilon_i\circ\theta,f\circ\theta:T'\to\mathbb{C}$ are all positive in the above sense.
(ii) For any $i\in I$, the rational map $e_{i,\theta}:\mathbb{C}^\times\times T'\to T'$ defined by $e_{i,\theta}(c,t):=\theta^{-1}\circ e_i^c\circ\theta(t)$ is positive.

Let $v : \mathbb{C}(c)\backslash \to \mathbb{Z}$ be a map defined by $v(f(c)) := \deg(f(c^{-1}))$, which is different from that in e.g., [11, 12, 15, 17].

Let $f\colon T \to T'$ be a positive rational mapping of algebraic tori T and T'. We define the map $\widehat{f}\colon X_*(T) \to X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi)\rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Let $\mathcal{T}_+$ be the category whose objects are algebraic tori over $\mathbb{C}$ and whose morphisms are positive rational maps. Then, we obtain the "*ultra-discretization*" functor

$$\begin{array}{cccc} \mathcal{UD} : & \mathcal{T}_+ & \longrightarrow & \mathfrak{S}et \\ & T & \mapsto & X_*(T) \\ & (f : T \to T') & \mapsto & (\widehat{f} : X_*(T) \to X_*(T')). \end{array}$$

Note that this definition of the functor $\mathcal{UD}$ is called tropicalization in [1] and much simpler than the one in [2].

Let $\theta : T' \to X$ be a positive structure on a decorated geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I}, f)$. Applying the functor $\mathcal{UD}$ to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T' \to T'$ and $f \circ \Theta, \gamma_i \circ \theta, \varepsilon_i \circ \Theta : T' \to \mathbb{C}$, we obtain

$$\begin{array}{rcl} \tilde{e}_i & := & \mathcal{UD}(e_{i,\theta}) : \mathbb{Z} \times X_*(T') \to X_*(T') \\ \mathrm{wt}_i & := & \mathcal{UD}(\gamma_i \circ \theta) : X_*(T') \to \mathbb{Z}, \\ \widetilde{\varepsilon}_i & := & \mathcal{UD}(\varepsilon_i \circ \theta) : X_*(T') \to \mathbb{Z}, \\ \widetilde{f} & := & \mathcal{UD}(f \circ \theta) : X_*(T') \to \mathbb{Z}. \end{array}$$

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$, we associate the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\widetilde{\varepsilon}_i\}_{i\in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $\mathcal{UD}_{\theta,T'}(\chi)$. We have the following theorem:

Theorem 4.4 ([1, 2, 15])**.** *For any geometric crystal $\chi = (X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{UD}_{\theta,T'}(\chi) = (X_*(T'), \{e_i\}_{i\in I}, \{\mathrm{wt}_i\}_{i\in I}, \{\widetilde{\varepsilon}_i\}_{i\in I})$ is a Langlands dual Kashiwara's crystal.*

Remark. The definition of $\widetilde{\varepsilon}_i$ is different from the one in [2, 6.1.] since our definition of ε_i corresponds to ε_i^{-1} in [2].

For a positive decorated geometric crystal $\mathcal{X} = ((X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I}, f), \theta, T')$, set

$$\widetilde{B}_{\widetilde{f}} := \{\widetilde{x} \in X_*(T') (= \mathbb{Z}^{\dim(T')}) | \widetilde{f}(\widetilde{x}) \geq 0\}, \tag{4.3}$$

and define $B_{f,\theta} := (\widetilde{B}_{\widetilde{f}}, \mathrm{wt}_i|_{\widetilde{B}_{\widetilde{f}}}, \varepsilon_i|_{\widetilde{B}_{\widetilde{f}}}, e_i|_{\widetilde{B}_{\widetilde{f}}})_{i\in I}$.

Proposition 4.5 ([2])**.** *For a positive decorated geometric crystal $\mathcal{X} = ((X, \{e_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I}, f), \theta, T')$, the quadruple $B_{f,\theta}$ is a normal crystal.*

4.4. **Decorated geometric crystal on $\mathbb{B}_w$.** For a Weyl group element $w \in W$, define $B_w^- := B^- \cap U\overline{w}U$ and set $\mathbb{B}_w := TB_w^-$. Let $\gamma_i : \mathbb{B}_w \to \mathbb{C}$ be the rational function defined by

$$\gamma_i : \mathbb{B}_w \hookrightarrow B^- \xrightarrow{\sim} T \times U^- \xrightarrow{\mathrm{proj}} T \xrightarrow{\alpha_i^\vee} \mathbb{C}. \tag{4.4}$$

For any $i \in I$, there exists the natural projection $pr_i : B^- \to B^- \cap \phi(SL_2)$. Hence, for any $x \in \mathbb{B}_w$ there exists unique $v = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \in SL_2$ such that $pr_i(x) = \phi_i(v)$. Using this fact, we define the rational function ε_i on $\mathbb{B}_w$ as in [18]:

$$\varepsilon_i(x) = \frac{b_{22}}{b_{21}} \quad (x \in \mathbb{B}_w). \tag{4.5}$$

The rational $\mathbb{C}^\times$-action e_i on $\mathbb{B}_w$ is defined by

$$e_i^c(x) := x_i\left((c-1)\varphi_i(x)\right)\cdot x\cdot x_i\left((c^{-1}-1)\varepsilon_i(x)\right) \qquad (c\in\mathbb{C}^\times,\ x\in\mathbb{B}_w), \tag{4.6}$$

if $\varepsilon_i(x)$ is well-defined, that is, $b_{21}\neq 0$, and define $e_i^c(x)=x$ if $b_{21}=0$.

Remark. The definition (4.5) is different from the one in [2]. Indeed, if in (4.5) we take $\varepsilon_i(x)=b_{21}/b_{22}$, then it coincides with the one in [2].

Proposition 4.6 ([2]). *For any $w\in W$, the 5-tuple $\chi := (\mathbb{B}_w, \{e_i\}_i, \{\gamma_i\}_i, \{\varepsilon_i\}_i, f_B)$ is a decorated geometric crystal, where f_B is in (4.2), γ_i is in (4.4), ε_i is in (4.5) and e_i is in (4.6).*

For the longest Weyl group element $w_0\in W$, let $\mathbf{i_0} = i_1\dots i_N$ be one of its reduced expressions and define the positive structure on $B^-_{w_0}$ $\Theta^-_{\mathbf{i_0}} : (\mathbb{C}^\times)^N \longrightarrow B^-_{w_0}$ by

$$\Theta^-_{\mathbf{i_0}}(c_1,\cdots,c_N) := \boldsymbol{y}_{i_1}(c_1)\cdots\boldsymbol{y}_{i_N}(c_N),$$

where $\boldsymbol{y}_i(c)=y_i(c)\alpha^\vee(c^{-1})$, which is different from $Y_i(c)$ in [15, 14, 11, 12]. Indeed, $Y_i(c)=\boldsymbol{y}_i(c^{-1})$. We also define the positive structure on $\mathbb{B}_{w_0}$ as $T\Theta^-_{\mathbf{i_0}} : T\times(\mathbb{C}^\times)^N \longrightarrow \mathbb{B}_{w_0}$ by $T\Theta^-_{\mathbf{i_0}}(t,c_1,\cdots,c_N) = t\Theta^-_{\mathbf{i_0}}(c_1,\cdots,c_N)$.

Now, for this positive structure, we describe the geometric crystal structure on $\mathbb{B}_{w_0} = TB^-_{w_0}$ explicitly.

Proposition 4.7 ([18]). *The action e_i^c on $t\Theta^-_{\mathbf{i_0}}(c_1,\cdots,c_N)$ is given by*

$$e_i^c(t\Theta^-_{\mathbf{i_0}}(c_1,\cdots,c_N)) = t\Theta^-_{\mathbf{i_0}}(c'_1,\cdots,c'_N)$$

where

$$c'_j := c_j\cdot\frac{\displaystyle\sum_{1\le m<j,\, i_m=i} c\cdot c_1^{\mathbf{a}_{i_1,i}}\cdots c_{m-1}^{\mathbf{a}_{i_{m-1},i}}c_m + \sum_{j\le m\le N,\, i_m=i} c_1^{\mathbf{a}_{i_1,i}}\cdots c_{m-1}^{\mathbf{a}_{i_{m-1},i}}c_m}{\displaystyle\sum_{1\le m\le j,\, i_m=i} c\cdot c_1^{\mathbf{a}_{i_1,i}}\cdots c_{m-1}^{\mathbf{a}_{i_{m-1},i}}c_m + \sum_{j< m\le N,\, i_m=i} c_1^{\mathbf{a}_{i_1,i}}\cdots c_{m-1}^{\mathbf{a}_{i_{m-1},i}}c_m}. \tag{4.7}$$

The explicit forms of rational functions ε_i and γ_i are:

$$\varepsilon_i(t\Theta^-_{\mathbf{i_0}}(c)) = \left(\sum_{1\le m\le N,\, i_m=i}\frac{1}{c_m c_{m+1}^{\mathbf{a}_{i_{m+1},i}}\cdots c_N^{\mathbf{a}_{i_N,i}}}\right)^{-1}, \quad \gamma_i(t\Theta^-_{\mathbf{i_0}}(c)) = \frac{\alpha_i(t)}{c_1^{\mathbf{a}_{i_1,i}}\cdots c_N^{\mathbf{a}_{i_N,i}}}. \tag{4.8}$$

5. Monomial Realization of Crystals

5.1. **Definitions of Monomial Realization of Crystals.** Following [9, 13], we shall introduce the monomial realization of crystals. For doubly-indexed variables $\{Y_{m,i} | i\in I, m\in\mathbb{Z}.\}$, define the set of monomials

$$\mathcal{Y} := \{Y = \prod_{m\in\mathbb{Z}, i\in I} Y_{m,i}^{l_{m,i}} | l_{m,i}\in\mathbb{Z}\setminus\{0\} \text{ except for finitely many } (m,i)\}.$$

Fix a set of integers $p=(p_{i,j})_{i,j\in I, i\neq j}$ such that $p_{i,j}+p_{j,i}=1$, which we call a *sign*. Take a sign $p:=(p_{i,j})_{i,j\in I,i\neq j}$ and a Cartan matrix $(\mathbf{a}_{i,j})_{i,j\in I}$. For $m\in\mathbb{Z}$, $i\in I$ define the monomial

$$A_{m,i} = Y_{m,i}Y_{m+1,i}\prod_{j\neq i} Y_{m+p_{j,i},j}^{\mathbf{a}_{j,i}}.$$

Here, when we emphasize the sign p, we shall denote the monomial $A_{m,i}$ by $A_{m,i}^{(p)}$. For any cyclic sequence of the indices $\iota = \cdots(i_1 i_2\cdots i_n)(i_1 i_2\cdots i_n)\cdots$ s.t. $\{i_1,\cdots,i_n\} = I$, we can associate the

following sign $(p_{i,j})$ by:

$$p_{i_a,i_b}=\begin{cases}1 & a<b,\\ 0 & a>b.\end{cases} \tag{5.1}$$

For example, if we take $\iota=\cdots(213)(213)\cdots$, then we have $p_{2,1}=p_{1,3}=p_{2,3}=1$ and $p_{1,2}=p_{3,1}=p_{3,2}=0$. Thus, we can identify a cyclic sequence $\cdots(i_1\cdots i_n)(i_1\cdots i_n)\cdots$ with such $(p_{i,j})$.

For a monomial $Y=\prod_{m,i}Y_{m,i}^{l_{m,i}}$, set

$$wt(Y)=\sum_{i,m}l_{m,i}\Lambda_i,\ \varphi_i(Y)=\max_{m\in\mathbb{Z}}\{\sum_{k\le m}l_{k,i}\},\ \varepsilon_i(Y)=\varphi_i(Y)-wt(Y)(h_i)=\max_{m\in\mathbb{Z}}\{-\sum_{k>m}l_{i,k}\},$$

$$\tilde{f}_i(Y)=\begin{cases}A^{-1}_{n_f^{(i)}(Y),i}\cdot Y & \text{if } \varphi_i(Y)>0,\\ 0 & \text{if } \varphi_i(Y)=0,\end{cases}\qquad \tilde{e}_i(Y)=\begin{cases}A_{n_e^{(i)}(Y),i}\cdot Y & \text{if } \varepsilon_i(Y)>0,\\ 0 & \text{if } \varepsilon_i(Y)=0,\end{cases}$$

$$\text{where } n_f^{(i)}(Y)=\min\{n|\varphi_i(Y)=\sum_{k\le n}l_{k,i}\},\quad n_e^{(i)}(Y)=\max\{n|\varphi_i(Y)=\sum_{k\le n}l_{k,i}\}.$$

Theorem 5.1 ([9, 13])**.** (*i*) *In the above setting, $\mathcal{Y}$ is a crystal, which is denoted by $\mathcal{Y}(p)$.*

(*ii*) *If $Y\in\mathcal{Y}(p)$ satisfies $\varepsilon_i(Y)=0$ (resp. $\varphi_i(Y)=0$) for any $i\in I$, then the connected component containing Y is isomorphic to $B(wt(Y))$ (resp. $B(w_0\mathrm{wt}(Y))$), where we call such monomial Y a highest (resp. lowest) monomial.*

By the above crystal structure of monomials, we know that for any $k\in\mathbb{Z}$, $i\in I$ the monomial $Y_{k,i}$ is a highest monomial with the weight Λ_i. Thus, we can define the embedding of crystal $m_i^{(k)}$ $(i\in I,k\in\mathbb{Z})$ by

$$\begin{array}{rcl} m_i^{(k)}:B(\Lambda_i) & \hookrightarrow & \mathcal{Y}(p)\\ u_{\Lambda_i} & \mapsto & Y_{k,i}\end{array} \tag{5.2}$$

5.2. **Duality on Monomial Realizations.** Let $i_1,\cdots,i_n$ be the indices satisfying $\{i_1,\cdots,i_n\}=I$. For the cyclic sequence $\mathbf{i}=\cdots(i_1i_2\cdots i_n)(i_1i_2\cdots i_n)\cdots$ let $p=(p_{i,j})$ be the associated sign as in(5.1).

And for the opposite cyclic sequence $\mathbf{i}^{-1}=\cdots(i_ni_{n-1}\cdots i_1)(i_ni_{n-1}\cdots i_1)\cdots$, let ${}^tp=p':=(p'_{i,j})=(p_{j,i})$ be the associated sign. For a monomial $Y=\prod_{m,i}Y_{i,m}^{l_{i,m}}\in\mathcal{Y}(p)$, define the map ${}^{a}-:\mathcal{Y}(p)\to\mathcal{Y}({}^tp)$ is defined by $Y_{i,m}\mapsto Y^{-1}_{i,a-m}$ for $a\in\mathbb{Z}$.

Indeed, the following lemma is derived by direct calculations:

Proposition 5.2. *For a monomial $Y\in\mathcal{Y}(p)$, we have ${}^a\overline{Y}\in\mathcal{Y}({}^tp)$ and*

$$\varphi_i({}^a\overline{Y})=\varepsilon_i(Y),\quad \varepsilon_i({}^a\overline{Y})=\varphi_i(Y),\quad wt({}^a\overline{Y})=-wt(Y), \tag{5.3}$$

$${}^a\overline{\tilde{f}_i(Y)}=\tilde{e}_i({}^a\overline{Y}),\qquad {}^a\overline{\tilde{e}_i(Y)}=\tilde{f}_i({}^a\overline{Y}). \tag{5.4}$$

Proof. The last formula in (5.3) is trivial by the definition. For an arbitrary monomial $Y=\prod_{i,m}Y_{i,m}^{l_{i,m}}\in\mathcal{Y}(p)$, we have ${}^a\overline{Y}=\prod_{i,m}Y_{i,a-m}^{-l_{i,m}}=\prod_{i,m}Y_{i,m}^{-l_{i,a-m}}$. Thus, one gets

$$\varphi_i({}^a\overline{Y})=\max_{m\in\mathbb{Z}}\{\sum_{k\le m}-l_{i,a-k}\}=\max_{m\in\mathbb{Z}}\{\sum_{a-k\le m}-l_{i,k}\}=\max_{m\in\mathbb{Z}}\{\sum_{k\ge m}-l_{i,k}\}=\varepsilon_i(Y).$$

The second one is obtained similarly.

As for (5.4), we shall show the first one. Let us see the following lemma:

Lemma 5.3. *We have*

$$ {}^a\overline{A^{(p)}_{i,m}(Y)} = A^{({}^t p)}_{i,a-m-1}(Y), \tag{5.5}$$

$$ n^{(i)}_e({}^a\overline{Y}) = a - n^{(i)}_f(Y) - 1, \quad n^{(i)}_f({}^a\overline{Y}) = a - n^{(i)}_e(Y) - 1, \tag{5.6}$$

Proof. The first formula is trivial from the definition of $A_{i,m}$. As for (5.6), we can see

$$\begin{aligned} n^{(i)}_e({}^a\overline{Y}) &= \max_n\{\varphi_i({}^a\overline{Y}) = \sum_{k\le n} -l_{i,a-k}\} = \max_n\{\varepsilon_i(Y) = \sum_{k\le n} -l_{i,a-k}\} \\ &= \max_n\{\mathrm{wt}_i(Y) + \varepsilon_i(Y) = \mathrm{wt}_i(Y) + \sum_{k\le n} -l_{i,a-k}\} = \max_n\{\varphi_i(Y) = \sum_{k>n} l_{i,a-k}\} \\ &= \max_n\{\varphi_i(Y) = \sum_{k<a-n} l_{i,k}\} = a - \min_n\{\varphi_i(Y) = \sum_{k<n} l_{i,k}\} = a - (n^{(i)}_f(Y) + 1). \end{aligned}$$

The second one is shown similarly. □

By this lemma, we can easily get (5.4). □

For $Y \in \mathcal{Y}(p)$, let us denote the connected component containing Y by $B(Y)$. Then, by the above proposition we obtain:

Theorem 5.4. *For any $Y \in \mathcal{Y}(p)$ and any $a \in \mathbb{Z}$, the set ${}^a\overline{B(Y)}$ is equipped with the crystal structure associated with ${}^t p$ and there exists the isomorphism of crystals :*

$$ {}^a\overline{B(Y)} \xrightarrow{\sim} B({}^a\overline{Y}) (\subset \mathcal{Y}({}^t p)) \qquad ({}^a\overline{Y} \mapsto {}^a\overline{Y}). \tag{5.7}$$

Indeed, we find that if Y is a highest(resp. lowest) monomial in $\mathcal{Y}(p)$, then ${}^a\overline{Y}$ is a lowest(resp. highest) monomial in $\mathcal{Y}({}^t p)$.

6. Explicit form of the decoration f_B for Classical Groups

6.1. **Generalized Minors and the function f_B.** For this subsection, see [3, 4, 5]. Let G be a simply connected simple algebraic groups over $\mathbb{C}$ and $T \subset G$ a maximal torus. Let $X^*(T) := \mathrm{Hom}(T, \mathbb{C}^\times)$ and $X_*(T) := \mathrm{Hom}(\mathbb{C}^\times, T)$ be the lattice of characters and co-characters respectively. We identify P (resp. P^*) with $X^*(T)$ (resp. $X_*(T)$) as above.

Definition 6.1. For a dominant weight $\mu \in P_+$, the *principal minor* $\Delta_\mu : G \to \mathbb{C}$ is defined as

$$\Delta_\mu(u^- t u^+) := \mu(t) \quad (u^\pm \in U^\pm,\ t \in T).$$

Let $\gamma, \delta \in P$ be extremal weights such that $\gamma = u\mu$ and $\delta = v\mu$ for some $u, v \in W$. Then the *generalized minor* $\Delta_{\gamma,\delta}$ is defined by

$$\Delta_{\gamma,\delta}(g) := \Delta_\mu(\overline{u}^{-1} g \overline{v}) \quad (g \in G),$$

which is a regular function on G.

Lemma 6.2 ([2])**.** *Suppose that G is simply connected.*

(*i*) *For $u \in U$ and $i \in I$, we have $\Delta_{\mu,\mu}(u) = 1$ and $\chi_i(u) = \Delta_{\Lambda_i, s_i\Lambda_i}(u)$, where Λ_i be the ith fundamental weight.*

(*ii*) *Define the map $\pi^+ : B^- \cdot U \to U$ by $\pi^+(bu) = u$ for $b \in B^-$ and $u \in U$. For any $g \in G$, we have*

$$\chi_i(\pi^+(g)) = \frac{\Delta_{\Lambda_i, s_i\Lambda_i}(g)}{\Delta_{\Lambda_i,\Lambda_i}(g)}. \tag{6.1}$$

Proposition 6.3 ([2])**.** *The function f_B in (4.2) is described as follows:*

$$ f_B(g) = \sum_i \frac{\Delta_{w_0\Lambda_i, s_i\Lambda_i}(g) + \Delta_{w_0 s_i\Lambda_i,\Lambda_i}(g)}{\Delta_{w_0\Lambda_i,\Lambda_i}(g)}. \tag{6.2}$$

6.2. **Bilinear Forms.** Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the anti involution

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i \quad \omega(h) = h,$$

and extend it to G by setting $\omega(x_i(c)) = y_i(c)$, $\omega(y_i(c)) = x_i(c)$ and $\omega(t) = t$ $(t \in T)$.

There exists a $\mathfrak{g}$(or G)-invariant bilinear form on the finite-dimensional irreducible $\mathfrak{g}$-module $V(\lambda)$ such that

$$\langle au, v\rangle = \langle u, \omega(a)v\rangle, \qquad (u, v \in V(\lambda),\ a \in \mathfrak{g}(\text{or } G)).$$

For $g \in G$, we have the following simple fact:

$$\Delta_{\Lambda_i}(g) = \langle gu_{\Lambda_i}, u_{\Lambda_i}\rangle,$$

where u_{Λ_i} is a properly normalized highest weight vector in $V(\Lambda_i)$. Hence, for $w, w' \in W$ we have

$$\Delta_{w\Lambda_i, w'\Lambda_i}(g) = \Delta_{\Lambda_i}(\overline{w}^{-1} g \overline{w}') = \langle \overline{w}^{-1} g \overline{w}' \cdot u_{\Lambda_i}, u_{\Lambda_i}\rangle = \langle g\overline{w}' \cdot u_{\Lambda_i}\ ,\ \overline{w} \cdot u_{\Lambda_i}\rangle, \tag{6.3}$$

where note that $\omega(\overline{s}_i^{\pm}) = \overline{s}_i^{\mp}$.

Proposition 6.4. *Let $\mathbf{i} = i_1 \cdots i_N$ be a reduced word for the longest Weyl group element w_0. For $t\Theta_{\mathbf{i}}^-(c) \in \mathbb{B}_{w_0} = T \cdot B_{w_0}^-$, we get the following formula.*

$$f_B(t\Theta_{\mathbf{i}}^-(c)) = \sum_i \Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i}}^-(c)) + \alpha_i(t)\Delta_{w_0 s_i\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)). \tag{6.4}$$

Proof. We shall show

$$\Delta_{\overline{w}_0\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) = 1. \tag{6.5}$$

Since $\Theta_{\mathbf{i}}^-(c) \in U\overline{w}_0 U$, we have $\overline{w}_0^{-1}\Theta_{\mathbf{i}}^-(c) \in \overline{w}_0^{-1} U \overline{w}_0 U = U^- \cdot U$. So, there exist $u_1 \in U^-$ and $u_2 \in U$ such that $\overline{w}_0^{-1}\Theta_{\mathbf{i}}^-(c) = u_1 u_2$. Thus, it follows from (6.3) that

$$\Delta_{\overline{w}_0\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) = \langle \overline{w}_0^{-1}\Theta_{\mathbf{i}}^-(c) u_{\Lambda_i}, u_{\Lambda_i}\rangle = \langle u_1 u_2 u_{\Lambda_i}, u_{\Lambda_i}\rangle = \langle u_2 u_{\Lambda_i}, \omega(u_1) u_{\Lambda_i}\rangle = \langle u_{\Lambda_i}, u_{\Lambda_i}\rangle = 1,$$

since $\omega(u_1) \in U$. The following is evident from (6.3)

$$\begin{aligned} \Delta_{w_0 s_i\Lambda_i, \Lambda_i}(t\Theta_{\mathbf{i}}^-(c)) &= \langle t\Theta_{\mathbf{i}}^-(c) u_{\Lambda_i}, \overline{w}_0 \overline{s}_i u_{\Lambda_i}\rangle = \langle \Theta_{\mathbf{i}}^-(c) u_{\Lambda_i}, t \cdot \overline{w}_0 \overline{s}_i u_{\Lambda_i}\rangle \\ &= \Delta_{w_0 s_i\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) \cdot w_0 s_i \Lambda_i(t). \end{aligned} \tag{6.6}$$

Since $w_0 s_i \Lambda_i / w_0 \Lambda_i = \alpha_i$ and $\Delta_{w_0\Lambda_i, \Lambda_i}(t\Theta_{\mathbf{i}}^-(c)) = \Delta_{w_0\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) \cdot w_0\Lambda_i(t)$, we obtain the desired result. □

Remark. Note that by virtue of Proposition 6.4 to get the explicit form of $f_B(t\Theta_{\mathbf{i}}^-(c))$ it is sufficient to know those of $\Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i}}^-(c))$ and $\Delta_{w_0 s_i\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c))$.

6.3. **Explicit form of $f_B(t\Theta_{\mathbf{i}}^-(c))$ for A_n.** For all classical cases A_n, B_n, C_n and D_n, we fix the reduced longest word $\mathbf{i_0}$ as follows:

$$\mathbf{i_0} = \begin{cases} \underbrace{1, 2, \cdots, n}, \underbrace{1, 2, \cdots, n-1}, \cdots, \underbrace{1, 2, 3}, 1, 2, 1 & \text{for } A_n, \\ (1, 2, \cdots, n-1, n)^n & \text{for } B_n,\ C_n, \\ (1, 2, \cdots, n-1, n)^{n-1} & \text{for } D_n. \end{cases} \tag{6.7}$$

To obtain the explicit form of $f_B(t\Theta_{\mathbf{i_0}}^-(c))$ for type A_n, by the above remark it suffices to know $\Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i_0}}^-(c))$ and $\Delta_{w_0 s_i\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i_0}}^-(c))$ for

$$c = (c_j^{(i)} | i + j \le n+1) = (c_1^{(1)}, c_2^{(1)}, \cdots, c_n^{(1)}, c_1^{(2)}, c_2^{(2)}, \cdots, c_{n-1}^{(2)}, \cdots c_1^{(n-1)}, c_2^{(n-1)}, c_1^{(n)}) \in (\mathbb{C}^\times)^N.$$

The following result for type A_n is given in [18]:

Theorem 6.5 ([18]). *For $c \in (\mathbb{C}^\times)^N$ as above, we have the following explicit forms:*

$$\Delta_{w_0\Lambda_j,s_j\Lambda_j}(\Theta^-_{\mathbf{i_0}}(c)) = c_1^{(n-j+1)} + \frac{c_2^{(n-j+1)}}{c_1^{(n-j+2)}} + \frac{c_3^{(n-j+1)}}{c_2^{(n-j+2)}} + \cdots + \frac{c_j^{(n-j+1)}}{c_{j-1}^{(n-j+2)}}, \tag{6.8}$$

$$\Delta_{w_0s_j\Lambda_j,\Lambda_j}(\Theta^-_{\mathbf{i_0}}(c)) = \frac{1}{c_1^{(j)}} + \frac{c_1^{(j-1)}}{c_2^{(j-1)}} + \frac{c_2^{(j-2)}}{c_3^{(j-2)}} + \cdots + \frac{c_{j-1}^{(1)}}{c_j^{(1)}}, \ (j \in I). \tag{6.9}$$

As we mentioned in the introduction, we shall see the relations between the decoration and the monomial realizations of crystals of type A_n explicitly following [18]. For type A_n take $(p_{i,j})_{i,j\in I,i\neq j}$ such that $p_{i,j} = 1$ for $i < j$, $p_{i,j} = 0$ for $i > j$, which corresponds to the cyclic sequence $\mathbf{i} = (12\cdots n)(12\cdots n)\cdots$. Then we obtain

Proposition 6.6 ([18]). *The crystal containing the monomial $Y_{n-i+1,1}$ (resp. $Y^{-1}_{i,1}$) is isomorphic to $B(\Lambda_1)$ (resp. $B(\Lambda_n)$) and all basis vectors are given by*

$$\tilde{f}_k\cdots\tilde{f}_2\tilde{f}_1(Y_{n-i+1,1}) = \frac{Y_{n-i+1,k+1}}{Y_{n-i+2,k}} \in B(\Lambda_1),$$
$$\tilde{e}_k\cdots\tilde{e}_2\tilde{e}_1(Y^{-1}_{i,1}) = \frac{Y_{i-k,k}}{Y_{i-k,k+1}} \in B(\Lambda_n) \quad (k = 1,\cdots,n).$$

Applying this results to Theorem 6.5 and changing the variable $Y_{m,l}$ to $c_l^{(m)}$, we find:

Proposition 6.7 ([18]). *For $j = 1,\cdots,n$ we have*

$$\chi_j(\pi^+(w_0^{-1}t\Theta^-_{\mathbf{i_0}}(c))) = \Delta_{w_0\Lambda_j,s_j\Lambda_j}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{k=0}^{j-1}\tilde{f}_k\cdots\tilde{f}_2\tilde{f}_1(c_1^{(n-j+1)}),$$
$$\chi_j(\pi^+(w_0^{-1}\eta(t\Theta^-_{\mathbf{i_0}}(c)))) = \alpha_j(t)\Delta_{w_0s_j\Lambda_j,\Lambda_j}(\Theta^-_{\mathbf{i_0}}(c)) = \alpha_j(t)\sum_{k=0}^{j-1}\tilde{e}_k\cdots\tilde{e}_2\tilde{e}_1({c_1^{(j)}}^{-1}).$$

Note that $\{\tilde{f}_k\cdots\tilde{f}_2\tilde{f}_1(c_{n-i+1,1})|0 \le k < i\} = B(\Lambda_1)s_{k-1}\cdots s_2s_1$ is the Demazure crystal associated with the Weyl group element $s_{k-1}\cdots s_2s_1$ ([8]).

Observing Proposition 6.7, we present the following conjecture:

Conjecture 6.8 ([18]). *There exists certain reduced longest word $\mathbf{i} = (i_1,\cdots,i_N)$ and a sign $p = (p_{i,j})_{i\neq j}$ such that for any $i \in I$, there exist Demazure crystal $B^-_w(i) \subset B(\Lambda_k)$, Demazure crystal $B^+_{w'}(i) \subset B(\Lambda_j)$ and positive integers $\{a_b, a_{b'}|b \in B^-_w, b' \in B^+_{w'}\}$ satisfying*

$$\chi_i(\pi^+(w_0^{-1}t\Theta^-_{\mathbf{i}}(c))) = \Delta_{w_0\Lambda_i,s_i\Lambda_i}(\Theta^-_{\mathbf{i}}(c)) = \sum_{b\in B^-_w(i)} a_b m_b(c),$$
$$\chi_i(\pi^+(w_0^{-1}\eta(t\Theta^-_{\mathbf{i}}(c)))) = \alpha_i(t)\Delta_{w_0s_i\Lambda_i,\Lambda_i}(\Theta^-_{\mathbf{i}}(c)) = \alpha_i(t)\sum_{b'\in B^+_{w'}(i)} a_{b'} m_{b'}(c),$$

where $m_b(c) \in \mathcal{Y}(p)$ is the monomial corresponding to $b \in B(\Lambda_k)$ associated with $p = (p_{i,j})_{i\neq j}$.

We would see the answers to this conjecture for other type of Lie algebras in the subsequent sections.

7. Explicit form of $f_B(t\Theta^-_{\mathbf{i_0}}(c))$ for C_n

7.1. **Main theorems.** In this section we see the results for type C_n.

Theorem 7.1. *In the case C_n, for $k = 1, \cdots, n$, $\mathbf{i_0} = (12\cdots n)^n$ and $c = (c_i^{(j)})_{1\le i,j\le n}$ $= (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(n)}, c_n^{(n)}) \in (\mathbb{C}^\times)^{n^2}$ we have*

$$\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = c_1^{(k)} + \frac{c_2^{(k)}}{c_1^{(k+1)}} + \cdots + \frac{c_n^{(k)}}{c_{n-1}^{(k+1)}} + \frac{c_{n-1}^{(k+1)}}{c_n^{(k+1)}} + \frac{c_{n-2}^{(k+2)}}{c_{n-1}^{(k+2)}} + \cdots + \frac{c_k^{(n)}}{c_{k+1}^{(n)}}, \tag{7.1}$$

where note that $\Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = c_n^{(n)}$.

We also get the following theorem.

Theorem 7.2. (*i*) *Let k be in $\{1, 2, \cdots, n-1\}$. Then we have*

$$\Delta_{w_0s_k\Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \frac{1}{c_1^{(k)}} + \sum_{j=1}^{k-1} \frac{c_{k-j}^{(j)}}{c_{k-j+1}^{(j)}}. \tag{7.2}$$

(*ii*) *For variables $(c_i^{(j)})_{1\le i,j\le n}$, set $C_i^{(j)} = \frac{c_i^{(n-j)}}{c_{i-1}^{(n-j+1)}}$ and $\overline{C}_i^{(j)} = \frac{c_{i-1}^{(n-j)}}{c_i^{(n-j)}}$. Then we have*

$$\Delta_{w_0s_n\Lambda_n, \Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{(*)} C_{u_1}^{(1)} C_{u_2}^{(2)} \cdots C_{u_m}^{(m)} \overline{C}_{l_1}^{(l_1-1)} \cdot \overline{C}_{l_2}^{(l_2-2)} \cdots \overline{C}_{l_k}^{(l_k-k)}, \tag{7.3}$$

where $(*)$ is the conditions: $k + m = n$, $0 \le m < n$,
$1 \le l_1 < l_2 < \cdots < l_k \le n$, and $1 \le u_1 < u_2 < \cdots < u_m \le n$.

7.2. **Proof of Theorem 7.1.** On the module $V(\Lambda_1)$ we can write $\boldsymbol{x}_i(c) := \alpha_i^\vee(c^{-1})x_i(c) = c^{-h_i}(1 + c\cdot e_i)$ and $\boldsymbol{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) = (1 + c\cdot f_i)c^{-h_i}$ since $f_i^2 = e_i^2 = 0$ on $V(\Lambda_1)$.

We also have $\omega(\boldsymbol{y}_i(c)) = \alpha_i^\vee(c^{-1})x_i(c) = \boldsymbol{x}_i(c)$ and define ${}^{\overline{i}}\Xi_j^{(p)} = {}^{\overline{i}}\Xi_j^{(p)}(c^{[1:p]})$ and ${}^{i}\Xi_j^{(p)} = {}^{i}\Xi_j^{(p)}(c^{[1:p]})$ for $p, j \in I$, $c = (c_i^{(j)})_{1\le i,j\le n} \in (\mathbb{C}^\times)^{n^2}$ by

$$X^{(p)}X^{(p-1)}\cdots X^{(1)}v_i = \sum_{j=1}^n {}^{i}\Xi_j^{(p)} v_j + \sum_{j=1}^n {}^{i}\Xi_{\overline{j}}^{(p)} v_{\overline{j}} \in V(\Lambda_1) \quad (i = 1, 2, \cdots, n),$$
$$X^{(p)}X^{(p-1)}\cdots X^{(1)}v_{\overline{i}} = \sum_{j=1}^n {}^{\overline{i}}\Xi_j^{(p)} v_j + \sum_{j=1}^n {}^{\overline{i}}\Xi_{\overline{j}}^{(p)} v_{\overline{j}} \in V(\Lambda_1) \quad (i = 1, 2, \cdots, n),$$

where $c^{[1:p]} = (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(p)}, c_n^{(p)})$ and $X^{(p)} = \boldsymbol{x}_n(c_n^{(p)})\boldsymbol{x}_{n-1}(c_{n-1}^{(p)})\cdots \boldsymbol{x}_1(c_1^{(p)})$.

By (6.3) and $\omega(\Theta_{\mathbf{i}_0}(c)) = X^{(n)}\cdots X^{(1)}$, we have

$$\Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i_0}}(c)) = \langle \overline{s}_i \cdot u_{\Lambda_i}\, ,\, X^{(n)}\cdots X^{(1)}v_{\Lambda_i}\rangle, \tag{7.4}$$

and then e.g.,${}^{\overline{1}}\Xi_2^{(n)} = \Delta_{w_0\Lambda_1, s_1\Lambda_1}(\Theta_{\mathbf{i_0}}(c))$.

Here to describe ${}^{\overline{i}}\Xi_j^{(p)}$ explicitly let us introduce some combinatorial object *segments* as follows. For $1 \le p, k \le n$ set $L := p - n + k$ and $S := n - k + 1$. For $r = 0, 1, \cdots, n-p$, set

$$\mathcal{M}_k^{(p)}[r] := \{M = \{m_2, m_3, \cdots, m_L\} | 2 + r \le m_2 < \cdots m_L \le p + r\}.$$

We usually denote $\mathcal{M}_k^{(p)}[0]$ by $\mathcal{M}_k^{(p)}$. Define the *segments* of $M \in \mathcal{M}_k^{(p)}[r]$ as $M = M_1 \sqcup \cdots \sqcup M_S$ where each segment M_j is a consecutive subsequence of M or an empty set such that $\min(M_b) = \max(M_a) + (b - a + 1)$ for $a < b$ and non-empty M_a, M_b. Note that for $M = \{m_2, \cdots, m_L\} \in \mathcal{M}_k^{(p)}[r]$ and $s \in \{-r, -r+1, \cdots, n-p-r\}$ we find that $M[s] := \{m_2 + s, \cdots, m_L + r\}$ is an element in $\mathcal{M}_k^{(p)}[r + s]$.

Example 7.3. *For $n = p = 6$, $k = 4$ and $r = 0$ we have $L = 4$ and $S = 3$.*
$M = \{2,3,5\} \Longrightarrow M_1 = \{2,3\}$, $M_2 = \{5\}$, $M_3 = \emptyset$.
$M = \{2,3,6\} \Longrightarrow M_1 = \{2,3\}$, $M_2 = \emptyset$, $M_3 = \{6\}$,
$M = \{2,4,6\} \Longrightarrow M_1 = \{2\}$, $M_2 = \{4\}$, $M_3 = \{6\}$,
$M = \{3,4,6\} \Longrightarrow M_1 = \emptyset$, $M_2 = \{3,4\}$, $M_3 = \{6\}$,

For $m \in M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}[r]$, define $n(m) := n-j+1$ if $m \in M_j$. For $M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}[r]$, write $M_1 = \{2+r, 3+r, \cdots, a\}$ where note that non-empty M_1 has to include $2+r$. For $i_{2+r}, \cdots, i_a$ satisfying $i-1 \leq i_{2+r} \leq i_{3+r} \leq \cdots \leq i_a \leq n$, define

$$C^M_{i_{2+r},i_{3+r},\cdots,i_a} := \frac{c^{(1+r+\epsilon_{i_{2+r}})}_{i_{2+r}+1-2\epsilon_{i_{2+r}}} \cdots c^{(a+\epsilon_{i_a}-1)}_{i_a+1-2\epsilon_{i_a}}}{c^{(2+r)}_{i_{2+r}} \cdots c^{(a)}_{i_a}}, \qquad D^M := \prod_{m \in M \setminus M_1} \frac{c^{(m)}_{n(m)-1}}{c^{(m)}_{n(m)}},$$

where $\epsilon_i = \delta_{i,n}$ and $C^M_{i_{2+r},i_{3+r},\cdots,i_a} = 1$ (resp. $D^M = 1$) if $M_1 = \emptyset$ (resp. $M \setminus M_1 = \emptyset$). Here, for $(c^{(l)}_k)$ we set $c^{(l)} := (c^{(l)}_1, \cdots, c^{(l)}_n)$ and $c^{[a:b]} := (c^{(a)}, c^{(a+1)}, \cdots, c^{(b)})$. Then, $c = c^{[1:n]}$. Indeed, for $M \in \mathcal{M}_k^{(p)}[r]$ the monomial $C^M \cdot D^M$ depends on $c^{[2+r:p+r]}$ and then for any $s = 1, 2, \cdots, n-p+2$ and $q = 0, 1, \cdots, r$ we have

$$C^M_{i_{2+r},i_{3+r},\cdots,i_a} \cdot D^M = C^M_{i_{2+r},i_{3+r},\cdots,i_a} \cdot D^M(c^{[s:s+p-2]}) = C^{M[-q]}_{i_{2+r},i_{3+r},\cdots,i_a} \cdot D^{M[-q]}(c^{[s:s+p-2]}), \tag{7.5}$$

where $M[-q] \in \mathcal{M}_k^{(p)}[r-q]$.

Proposition 7.4. *In the setting above, we have*

$$^{\bar{i}}\Xi^{(p)}_k = \frac{1}{c^{(1)}_{i-1}} \sum_{\substack{i-1 \leq i_2 \leq \cdots \leq i_a \leq n \\ M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}}} C^M_{i_2,i_3,\cdots,i_a} \cdot D^M, \tag{7.6}$$

$$^{\bar{i}}\Xi^{(p)}_{\bar{k}} = \sum_{i=i_1 \leq i_2 \leq \cdots \leq i_p \leq k} (c^{(1)}_{i_1-1} c^{(2)}_{i_2-1} \cdots c^{(p)}_{i_p-1})^{-1} (c^{(1)}_{i_2} c^{(2)}_{i_3} \cdots c^{(p-1)}_{i_p} c^{(p)}_k). \tag{7.7}$$

Proof. Set $\mathcal{X} := \boldsymbol{x}_n(c_n) \cdots \boldsymbol{x}_1(c_1)$. By calculating directly we have the formula:

$$\mathcal{X} v_i = \begin{cases} c_1^{-1} v_1 & \text{if } i = 1, \\ c_{i-1} c_i^{-1} v_i + v_{i-1} & \text{if } i = 2, \cdots, n, \end{cases} \tag{7.8}$$

$$\mathcal{X} v_{\bar{i}} = c^{-1}_{i-1}(c_i v_{\bar{i}} + c_{i+1} v_{\overline{i+1}} + \cdots + c_{n-1} v_{\overline{n-1}} + c_n v_{\bar{n}} + v_n), \tag{7.9}$$

where we understand $c_0 = 1$. Using these, for $k = 1, 2, \cdots, n$ and $p = 2, \cdots, n$ we get

$$^{\bar{i}}\Xi^{(p)}_{\bar{k}} = \sum_{j=i}^{k} {}^{\bar{i}}\Xi^{(p-1)}_{\bar{j}} \frac{c^{(p)}_k}{c^{(p)}_{j-1}}, \tag{7.10}$$

$$^{\bar{i}}\Xi^{(p)}_k = {}^{\bar{i}}\Xi^{(p-1)}_{k+1} + {}^{\bar{i}}\Xi^{(p-1)}_k \frac{c^{(p)}_{k-1}}{c^{(p)}_k}. \tag{7.11}$$

Indeed, the formula (7.7) is easily shown by the induction on p using (7.10).

To obtain (7.6) we see the segments of elements in $\mathcal{M}_k^{(p)}$, $\mathcal{M}_{k+1}^{(p-1)}$ and $\mathcal{M}_k^{(p-1)}$. $M = M_1 \sqcup \cdots \sqcup M_{n-k} \in \mathcal{M}_{k+1}^{(p-1)}$ can be seen as an element in $\mathcal{M}_k^{(p)}$ by setting $M_{n-k+1} = \emptyset$. For any $M' = M'_1 \sqcup \cdots \sqcup M'_{n-k+1} \in \mathcal{M}_k^{(p-1)}$, the last segment M'_{n-k+1} is empty or includes $p-1$. Indeed, the following lemma insures this fact:

Lemma 7.5. *For any* $M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}$ *and any* $i = 1, \cdots, S := n-k+1$ *we have*

$$\min M_i \geq i+1, \qquad \max M_i \leq L+i-1. \tag{7.12}$$

The proof of this lemma is done by the induction on p.
Thus, in any case if we set $M''_{n-k+1} = M'_{n-k+1} \cup \{p\}$, then $M'' := M'_1 \sqcup \cdots \sqcup M'_{n-k} \sqcup M''_{n-k+1}$ turns out to be an element in $\mathcal{M}_k^{(p)}$ and we have

$$C^{M''}_{i_2,\cdots,i_a} \cdot D^{M''} = C^{M'}_{i_2,\cdots,i_a} \cdot D^{M'} \frac{c^{(p)}_{k-1}}{c^{(p)}_k}.$$

Using this formula and (7.11), we obtain (7.6). □

Thus, for example, we have $\Delta_{w_0\Lambda_1, s_1\Lambda_1}(\Theta^-_{\mathbf{i_0}}(c)) = {}^{\overline{1}}\Xi^{(n)}_2 = \sum_{j=1}^n \frac{c^{(1)}_j}{c^{(2)}_{j-1}} + \sum_{j=2}^n \frac{c^{(j)}_{n-j+1}}{c^{(j)}_{n-j+2}}$.

To get the explicit form of $\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))$, we show the following lemma:

Lemma 7.6. *For* $k = 1, \cdots, n-1$ *set*

$$W_k := \begin{pmatrix} {}^{\overline{1}}\Xi^{(n)}_{k+1} & {}^{\overline{1}}\Xi^{(n)}_{k-1} & \cdots & {}^{\overline{1}}\Xi^{(n)}_{2} & {}^{\overline{1}}\Xi^{(n)}_{1} \\ {}^{\overline{2}}\Xi^{(n)}_{k+1} & {}^{\overline{2}}\Xi^{(n)}_{k-1} & \cdots & {}^{\overline{2}}\Xi^{(n)}_{2} & {}^{\overline{2}}\Xi^{(n)}_{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\overline{k}}\Xi^{(n)}_{k+1} & {}^{\overline{k}}\Xi^{(n)}_{k-1} & \cdots & {}^{\overline{k}}\Xi^{(n)}_{2} & {}^{\overline{k}}\Xi^{(n)}_{1} \end{pmatrix}. \tag{7.13}$$

Then, we have $\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \det W_k$.

Proof. As has been given in (3.4), for the lowest weight vector $v_{\Lambda_k} \in V(\Lambda_k)$ and $X := X^{(n)} \cdots X^{(1)} \in G$ we have

$$Xv_{\Lambda_k} = \sum_{\sigma \in \mathfrak{S}_k} \mathrm{sgn}(\sigma) Xv_{\overline{\sigma(k)}} \otimes \cdots \otimes Xv_{\overline{\sigma(1)}}.$$

Here note that for the simple reflection $s_k \in W$,

$$\overline{s}_k(v_1 \otimes \cdots \otimes v_{k-1} \otimes v_k) = v_1 \otimes \cdots \otimes v_{k-1} \otimes v_{k+1}.$$

Thus, it follows from the formula (7.4) that $\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))$ coincides with the coefficient of the vector $v_1 \otimes \cdots \otimes v_{k-1} \otimes v_{k+1}$ in Xv_{Λ_k}, which completes the proof. □

The last column of W_k is just

$${}^t({}^{\overline{1}}\Xi^{(n)}_1, {}^{\overline{2}}\Xi^{(n)}_1, \cdots, {}^{\overline{k}}\Xi^{(n)}_1) = {}^t(1, {c^{(1)}_1}^{-1}, {c^{(1)}_2}^{-1}, \cdots, {c^{(1)}_{k-1}}^{-1}).$$

Considering the elementary transformations on W_k by (i-th row) $-\frac{c^{(1)}_i}{c^{(1)}_{i-1}} \times$($i+1$-th row) for $i = 1, \cdots, k-1$, the (i,j)-entry of the transformed matrix $\tilde{W}_k$ is as follows:

Lemma 7.7. *The* (i,j)*-entry* $(\tilde{W}_k)_{i,j}$ *is:*

$$(\tilde{W}_k)_{i,j} = \begin{cases} {}^{\overline{i}}\Xi^{(n-1)}_{k+1}(c^{[2:n]}) & \text{if } j = 1, \\ {}^{\overline{i}}\Xi^{(n-1)}_{k-j+1}(c^{[2:n]}) & \text{if } j > 1, \end{cases} \tag{7.14}$$

where for $(c^{(l)}_k)_{1\leq k,l\leq n}$ *we set* $c^{(l)} := (c^{(l)}_1, \cdots, c^{(l)}_n)$ *and* $c^{[a:b]} := (c^{(a)}, c^{(a+1)}, \cdots, c^{(b)})$ *for* $1 \leq a < b \leq n$*. Then,* $c = c^{[1:n]}$ *as above. Note that* ${}^{\overline{i}}\Xi^{(p)}_k(c)$ *depends only on* $c^{[1:p]}$.

Proof. We shall show

$$\overline{i}\Xi_j^{(n)}(c)-\frac{c_i^{(1)}}{c_{i-1}^{(1)}}\overline{i+1}\Xi_j^{(n)}(c)=\frac{c_i^{(1)}}{c_{i-1}^{(1)}}\overline{i}\Xi_j^{(n-1)}(c^{[2,n]}). \tag{7.15}$$

For $M=M_1\sqcup\cdots\sqcup M_{n-j+1}\in\mathcal{M}_j^{(n)}$ such that $M_1=\{2,3,\cdots,a\}$ is non-empty, let $M':=M\setminus\{2\}$ be an element in $\mathcal{M}_j^{(n-1)}[1]$. Then we have

$$C^M_{i-1,i_3,\cdots,i_a}\cdot D^M=\frac{c_i^{(1)}}{c_{i-1}^{(2)}}C^{M\setminus\{2\}}_{i_3,\cdots,i_a}\cdot D^{M\setminus\{2\}},$$

where $M\setminus\{2\}$ is considered as an element in $\mathcal{M}_j^{(n-1)}[1]$ and by (7.5) the left-hand side of (7.15) is written as

$$\frac{c_i^{(1)}}{c_{i-1}^{(1)}c_{i-1}^{(2)}}\sum_{\substack{i-1\le i_3\le\cdots\le i_a\le n\\ M'(=M\setminus\{2\})=M'_1\sqcup\cdots\sqcup M'_{n-j+1}\in\mathcal{M}_j^{(n-1)}[1]}}C^{M'}_{i_3,\cdots,i_a}\cdot D^{M'},$$

which shows (7.15). $\square$

Applying the above elementary transformations to the matrix W_k repeatedly, we have the following:

Corollary 7.8. *For k,j such that $1\le k<j\le n$, we get*

$$\overline{i}\Xi_j^{(n)}(c^{[k:n]})-\frac{c_i^{(1)}}{c_{i-1}^{(1)}}\overline{i+1}\Xi_j^{(n)}(c^{[k:n]})=\frac{c_i^{(1)}}{c_{i-1}^{(1)}}\overline{i}\Xi_j^{(n-1)}(c^{[k+1,n]}). \tag{7.16}$$

Then we find the following equalities of determinants:

$$\begin{aligned}
\det W_k &= \begin{vmatrix}
c_1^{(1)}\overline{1}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & c_1^{(1)}\overline{1}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & c_1^{(1)}\overline{1}\Xi_{2}^{(n-1)}(c^{[2:n]}) & 0\\
\frac{c_2^{(1)}}{c_1^{(1)}}\overline{2}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & \frac{c_2^{(1)}}{c_1^{(1)}}\overline{2}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & \frac{c_2^{(1)}}{c_1^{(1)}}\overline{2}\Xi_{2}^{(n-1)}(c^{[2:n]}) & 0\\
\vdots & \vdots & \cdots & \vdots & \vdots\\
\frac{c_{k-1}^{(1)}}{c_{k-2}^{(1)}}\overline{k-1}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & \frac{c_{k-1}^{(1)}}{c_{k-2}^{(1)}}\overline{k-1}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & \frac{c_{k-1}^{(1)}}{c_{k-2}^{(1)}}\overline{k-1}\Xi_{2}^{(n-1)}(c^{[2:n]}) & 0\\
\overline{k}\Xi_{k+1}^{(n)}(c^{[2:n]}) & \overline{k}\Xi_{k-1}^{(n)}(c^{[2:n]}) & \cdots & \overline{k}\Xi_{2}^{(n)}(c^{[2:n]}) & {c_{k-1}^{(1)}}^{-1}
\end{vmatrix}\\
&= \begin{vmatrix}
\overline{1}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & \overline{1}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & \overline{1}\Xi_{2}^{(n-1)}(c^{[2:n]})\\
\overline{2}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & \overline{2}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & \overline{2}\Xi_{2}^{(n-1)}(c^{[2:n]})\\
\vdots & \vdots & \cdots & \vdots\\
\overline{k-1}\Xi_{k+1}^{(n-1)}(c^{[2:n]}) & \overline{k-1}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & \overline{k-1}\Xi_{2}^{(n-1)}(c^{[2:n]})
\end{vmatrix}\\
&= \cdots\cdots\cdots\\
&= \begin{vmatrix}
\overline{1}\Xi_{k+1}^{(n-k+2)}(c^{[k-1:n]}) & \overline{1}\Xi_{k-1}^{(n-k+2)}(c^{[k-1:n]})\\
\overline{2}\Xi_{k+1}^{(n-k+2)}(c^{[k-1:n]}) & \overline{2}\Xi_{k-1}^{(n-k+2)}(c^{[k-1:n]})
\end{vmatrix} = \overline{1}\Xi_{k+1}^{(n-k+1)}(c^{[k:n]})
\end{aligned}$$

Thus, it follows from (7.6) that for $k=1,2,\cdots,n-1$

(7.17)

$$\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i}}(c))=\overline{1}\Xi_{k+1}^{(n-k+1)}(c^{[k:n]})=c_1^{(k)}+\frac{c_2^{(k)}}{c_1^{(k+1)}}+\cdots+\frac{c_n^{(k)}}{c_{n-1}^{(k+1)}}+\frac{c_{n-1}^{(k+1)}}{c_n^{(k+1)}}+\frac{c_{n-2}^{(k+2)}}{c_{n-1}^{(k+2)}}+\cdots+\frac{c_k^{(n)}}{c_{k+1}^{(n)}}$$

The case $k=n$ is easily obtained by the formula in [5, Sect.4]:

$$\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=c_1^{(n)}. \tag{7.18}$$

Now, the proof of Theorem 7.1 has been accomplished. □

7.3. **Proof of Theorem 7.2.** For $k = 1, 2, \cdots, n-1$, set

$$U_k := \begin{pmatrix} {}^{\overline{1}}\Xi_k^{(n)} & {}^{\overline{1}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{1}}\Xi_2^{(n)} & {}^{\overline{1}}\Xi_1^{(n)} \\ {}^{\overline{2}}\Xi_k^{(n)} & {}^{\overline{2}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{2}}\Xi_2^{(n)} & {}^{\overline{2}}\Xi_1^{(n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\overline{k-1}}\Xi_k^{(n)} & {}^{\overline{k-1}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{k-1}}\Xi_2^{(n)} & {}^{\overline{k-1}}\Xi_1^{(n)} \\ {}^{\overline{k+1}}\Xi_k^{(n)} & {}^{\overline{k+1}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{k+1}}\Xi_2^{(n)} & {}^{\overline{k+1}}\Xi_1^{(n)} \end{pmatrix}.$$

Since $\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta_{\mathbf{i_0}}(c)) = \langle u_{\Lambda_k}, X^{(n)} \cdots X^{(1)} \overline{s}_k v_{\Lambda_k} \rangle$, the function $\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta_{\mathbf{i_0}}(c))$ is given as the coefficient of the vector $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ in $X^{(n)} \cdots X^{(1)} \overline{s}_k v_{\Lambda_k}$. Thus, by the argument in the proof of Theorem 7.1, we obtain:

$$\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \det U_k \quad (k = 1, 2, \cdots, n-1). \tag{7.19}$$

Using the formula (7.15), we have for $j = 1, 2, \cdots, k$

$$ {}^{\overline{k-1}}\Xi_j^{(n)}(c) - \frac{c_k^{(1)}}{c_{k-2}^{(1)}} {}^{\overline{k+1}}\Xi_j^{(n)}(c) = \frac{c_{k-1}^{(1)}}{c_{k-2}^{(i)}} {}^{\overline{k-1}}\Xi_j^{(n-1)}(c^{[2:n]}) + \frac{c_k^{(1)}}{c_{k-2}^{(1)}} {}^{\overline{k}}\Xi_j^{(n-1)}(c^{[2:n]}). \tag{7.20}$$

Here, applying this formula to the above determinant, we have

$$\begin{aligned} &\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) \\ &= \frac{c_{k-1}^{(1)}}{c_k^{(1)}} \begin{vmatrix} {}^{\overline{1}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{1}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{1}}\Xi_2^{(n-1)}(c^{[2:n]}) \\ {}^{\overline{2}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{2}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{2}}\Xi_2^{(n-1)}(c^{[2:n]}) \\ \vdots & \vdots & \cdots & \vdots \\ {}^{\overline{k-1}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{k-1}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{k-1}}\Xi_2^{(n-1)}(c^{[2:n]}) \end{vmatrix} \\ &\quad + \begin{vmatrix} {}^{\overline{1}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{1}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{1}}\Xi_2^{(n-1)}(c^{[2:n]}) \\ {}^{\overline{2}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{2}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{2}}\Xi_2^{(n-1)}(c^{[2:n]}) \\ \vdots & \vdots & \cdots & \vdots \\ {}^{\overline{k-1}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{k-1}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{k-1}}\Xi_2^{(n-1)}(c^{[2:n]}) \\ {}^{\overline{k+1}}\Xi_k^{(n-1)}(c^{[2:n]}) & {}^{\overline{k+1}}\Xi_{k-1}^{(n-1)}(c^{[2:n]}) & \cdots & {}^{\overline{k+1}}\Xi_2^{(n)}(c^{[2:n]}) \end{vmatrix}. \end{aligned}$$

In the above formula, let Z_{k-1} be the matrix in the first determinant. Thus, we have

$$\det U_k = \frac{c_{k-1}^{(1)}}{c_k^{(1)}} \det Z_{k-1} + \det U_{k-1}. \tag{7.21}$$

Repeating these steps, we can derive

$$\det U_k = \det U_1 + \sum_{j=1}^{k-1} \frac{c_{k-j}^{(j)}}{c_{k-j+1}^{(j)}} \det Z_j.$$

Carrying out the elementary transformations above to the matrix Z_{k-1} we easily know that $\det Z_j = 1$ for $j = 1, \cdots, k-1$ and $\det U_1 = {}^{\overline{2}}\Xi_k^{(n-k+1)}(c^{[k:n]}) = {c_1^{(k)}}^{-1}$, which show

$$\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \frac{1}{c_1^{(k)}} + \sum_{j=1}^{k-1} \frac{c_{k-j}^{(j)}}{c_{k-j+1}^{(j)}}, \tag{7.22}$$

and then (7.2).

Next, to show (7.3), we shall see $\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c))$ since for $g \in B^-_{w_0}$ we have

$$\Delta_{w_0 s_i\Lambda_i,\Lambda_i}(g) = \Delta_{w_0\Lambda_i,s_i\Lambda_i}(\eta(g)), \tag{7.23}$$

where $\mathbf{i_0}^{-1} = (n\, n-1 \cdots 21)^n$ and $\eta(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \Theta^-_{\mathbf{i_0}}(\overline{c})$ $(- : c_i^{(j)} \mapsto c^{(j)}_{n-i+1})$. These mean

$$\Delta_{w_0 s_i\Lambda_i,\Lambda_i}(\Theta^-_{\mathbf{i_0}}(c)) = \Delta_{w_0\Lambda_i,s_i\Lambda_i}(\Theta^-_{\mathbf{i_0}^{-1}}(c')),$$

where $c' = (c_N, c_{N-1}, \cdots, c_1)$ for $c = (c_1, \cdots, c_N)$. As we have seen $\omega(\boldsymbol{y}_i(c)) = \alpha_i^\vee(c^{-1})x_i(c) = \boldsymbol{x}_i(c)$, we have $\omega(\Theta^-_{\mathbf{i_0}^{-1}}(c')) = \overline{X}^{(n)}\overline{X}^{(n-1)}\cdots\overline{X}^{(1)}$ where $\overline{X}^{(p)} = \boldsymbol{x}_1(c_1^{(p)})\boldsymbol{x}_2(c_2^{(p)})\cdots\boldsymbol{x}_n(c_n^{(p)})$. We define ${}^{\overline{i}}\Sigma^{(p)}_j$ and ${}^{i}\Sigma^{(p)}_j$ by

$$\overline{X}^{(p)}\overline{X}^{(p-1)}\cdots\overline{X}^{(1)}v_{\overline{i}} = \sum_{j=1}^n {}^{\overline{i}}\Sigma^{(p)}_j v_j + \sum_{j=1}^n {}^{\overline{i}}\Sigma^{(p)}_{\overline{j}} v_{\overline{j}} \in V(\Lambda_1).$$

To describe these coefficients explicitly, we define the similar objects to the segments as above. For $1 \le p, i \le n$, set $L := p-n+i-1$ and $S := n-i+2$,

$$\overline{\mathcal{M}}^{(p)}_i := \{M = \{m_1, \cdots, m_L\} | 1 \le m_1 \le \cdots \le m_L \le p\}.$$

For an element M, define the segments $M_1, \cdots, M_S$ by the same way as before, thus, $M = M_1 \sqcup \cdots \sqcup M_S$. For an element $M = M_1 \sqcup \cdots \sqcup M_S \in \overline{\mathcal{M}}^{(p)}_i$ writing $M_S = \{q, q+1, \cdots, p-1, p\}$, set

$$D^M := \prod_{j=1}^S \prod_{m\in M_j} \frac{c^{(m)}_{i+j-1}}{c^{(m)}_{i+j-2}}, \qquad F^{M_S}_{j_q,j_{q+1},\cdots,j_p} := \frac{c^{(q-1)}_{j_q-1}c^{(q)}_{j_{q+1}-1}\cdots c^{(p-2)}_{j_{p-1}-1}c^{(p-1)}_{j_p-1}}{c^{(q)}_{j_q}c^{(q+1)}_{j_{q+1}}\cdots c^{(p-1)}_{j_{p-1}}c^{(p)}_{j_p}}.$$

Lemma 7.9. *We have*

$${}^{\overline{i}}\Sigma^{(p)}_k = c^{(p)}_{k-1} \sum_{\substack{k\,\le\, j_p\,\le\, j_{p-1}\cdots\,\le\, j_q\,\le\, n \\ M = M_1\sqcup\cdots\sqcup M_S \in \overline{\mathcal{M}}^{(p)}_i}} D^M \cdot F^{M_S}_{j_q,j_{q+1},\cdots,j_p}, \tag{7.24}$$

where $M_S = \{q, q+1, \cdots, p-1, p\}$ and $S = n-i+2$. Note that ${}^{\overline{1}}\Sigma^{(n)}_k = c^{(n)}_{k-1}$.

By the similar argument as the above cases, we have

$$\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \begin{vmatrix} {}^{\overline{1}}\Sigma^{(n)}_{\overline{n}} & {}^{\overline{1}}\Sigma^{(n)}_{n-1} & \cdots & {}^{\overline{1}}\Sigma^{(n)}_{2} & {}^{\overline{1}}\Sigma^{(n)}_{1} \\ {}^{\overline{2}}\Sigma^{(n)}_{\overline{n}} & {}^{\overline{2}}\Sigma^{(n)}_{n-1} & \cdots & {}^{\overline{2}}\Sigma^{(n)}_{2} & {}^{\overline{2}}\Sigma^{(n)}_{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\overline{n}}\Sigma^{(n)}_{\overline{n}} & {}^{\overline{n}}\Sigma^{(n)}_{n-1} & \cdots & {}^{\overline{n}}\Sigma^{(n)}_{2} & {}^{\overline{n}}\Sigma^{(n)}_{1} \end{vmatrix} \tag{7.25}$$

To calculate this determinant we need some preparations. Let us define the functions:

$${}^{\overline{i}}E_k[j](c) := \sum_{\substack{k\,\le\, j_k\,\le\, j_{k-1}\cdots\,\le\, j_q\,\le\, n \\ M = M_1\sqcup\cdots\sqcup M_S \in \overline{\mathcal{M}}^{(p)}_i, \\ M_S = \{q,q+1,\cdots,k\} \ne \emptyset, \\ j_k < u_{k+1} < u_{k+2}\cdots < u_j \le n}} D^M \cdot F^{M_S}_{j_q,j_{q+1},\cdots,j_k} \cdot F^{\{k+1,k+2,\cdots,j\}}_{u_{k+1},u_{k+2},\cdots,u_j} \quad (j \ge k), \tag{7.26}$$

$${}^{\overline{i}}F_k(c) := \sum_{\substack{M = \{m_1,\cdots,m_{k-n+i-1}\} \\ = M_1\sqcup\cdots\sqcup M_S \in \overline{\mathcal{M}}^{(p)}_i, \quad M_S = \emptyset, \\ 1\le l_1\le l_2\le\cdots\le l_{n-k+1}\le m_{k-n+i-1}}} D^M \cdot \frac{c^{(l_1)}_{l_1}c^{(l_2)}_{l_2+1}c^{(l_3)}_{l_3+2}\cdots c^{(l_{n-k+1})}_{l_{n-k+1}+n-k}}{c^{(l_1)}_{l_1-1}c^{(l_2)}_{l_2}c^{(l_3)}_{l_3+1}\cdots c^{(l_{n-k+1})}_{l_{n-k+1}+n-k-1}}. \tag{7.27}$$

Lemma 7.10. *We have the following formula for $2 \le k \le n$ and $2 \le p \le n$:*

$$ {}^{\bar{i}}\Sigma_k^{(p)} - \frac{c_{k-1}^{(p)}}{c_k^{(p)}} {}^{\bar{i}}\Sigma_{k+1}^{(p)} = \frac{c_{k-1}^{(p)}}{c_k^{(p)}} {}^{\bar{i}}\Sigma_k^{(p-1)}, \quad {}^{\bar{1}}\Sigma_k^{(n)} = c_{k-1}^{(n)}, \tag{7.28} $$

$$ {}^{\bar{i}}\Sigma_{n-1}^{(n)} - \frac{c_{n-2}^{(n)}}{{}^{\bar{1}}\Sigma_{\bar{n}}^{(n)}} \cdot {}^{\bar{i}}\Sigma_{\bar{n}}^{(n)} = \frac{c_{n-2}^{(n)}}{c_{n-1}^{(n)}} \cdot {}^{\bar{i}}\Sigma_{n-1}^{(n-1)} + c_{n-2}^{(n)} \cdot {}^{\bar{i}}E_n[n] + \frac{c_{n-2}^{(n)}}{{}^{\bar{1}}\Sigma_{\bar{n}}^{(n)}} \cdot {}^{\bar{i}}F_n, \tag{7.29} $$

$$ {}^{\overline{n-k+2}}F_k {}^{\bar{i}}\Sigma_{k-1}^{(k-1)} - c_{k-2}^{(k-1)} {}^{\bar{i}}F_k = c_{k-2}^{(k-1)} \cdot {}^{\overline{n-k+1}}F_k \cdot {}^{\bar{i}}E_{k-1}[k-1] + c_{k-2}^{(k-1)} {}^{\bar{i}}F_{k-1}, \tag{7.30} $$

$$ {}^{\overline{n-k+2}}E_k[j] \cdot {}^{\bar{i}}\Sigma_{k-1}^{(k-1)} - c_{k-2}^{(k-1)} \cdot {}^{\bar{i}}E_k[j] = c_{k-2}^{(k-1)} \cdot {}^{\bar{i}}E_{k-1}[j]. \tag{7.31} $$

Direct inspections show this lemma. □

Let us denote the right-hand side of (7.29) by X_i $(i = 2, \cdots, n)$. For the right-hand side of (7.25) subtracting the $\frac{c_{k-1}^{(n)}}{c_k^{(n)}} \times (n-k)$-th column from $n-k+1$th column for $k = 1, 2, \cdots, n-1$ and applying the formula (7.28) and (7.29), we obtain

(7.32) R.H.S of (7.25)

$$ = \frac{1}{c_{n-2}^{(n)}} \begin{vmatrix} {}^{\bar{1}}\Sigma_{\bar{n}}^{(n)} & 0 & \cdots & 0 & 0 \\ {}^{\bar{2}}\Sigma_{\bar{n}}^{(n)} & X_2 & \cdots & {}^{\bar{2}}\Sigma_2^{(n-1)} & {}^{\bar{2}}\Sigma_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}\Sigma_{\bar{n}}^{(n)} & X_n & \cdots & {}^{\bar{n}}\Sigma_2^{(n-1)} & {}^{\bar{n}}\Sigma_1^{(n-1)} \end{vmatrix} = \frac{{}^{\bar{1}}\Sigma_{\bar{n}}^{(n)}}{c_{n-1}^{(n)}} \begin{vmatrix} {}^{\bar{2}}\Sigma_{n-1}^{(n-1)} & \cdots & {}^{\bar{2}}\Sigma_2^{(n-1)} & {}^{\bar{2}}\Sigma_1^{(n-1)} \\ \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}\Sigma_{n-1}^{(n-1)} & \cdots & {}^{\bar{n}}\Sigma_2^{(n-1)} & {}^{\bar{n}}\Sigma_1^{(n-1)} \end{vmatrix} $$

$$ + {}^{\bar{1}}\Sigma_{\bar{n}}^{(n)} \begin{vmatrix} {}^{\bar{2}}E_n[n] & {}^{\bar{2}}\Sigma_{n-2}^{(n-1)} & \cdots & {}^{\bar{2}}\Sigma_2^{(n-1)} & {}^{\bar{2}}\Sigma_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}E_n[n] & {}^{\bar{n}}\Sigma_{n-2}^{(n-1)} & \cdots & {}^{\bar{n}}\Sigma_2^{(n-1)} & {}^{\bar{n}}\Sigma_1^{(n-1)} \end{vmatrix} + \begin{vmatrix} {}^{\bar{2}}F_n & {}^{\bar{2}}\Sigma_{n-2}^{(n-1)} & \cdots & {}^{\bar{2}}\Sigma_2^{(n-1)} & {}^{\bar{2}}\Sigma_1^{(n-1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}F_n & {}^{\bar{n}}\Sigma_{n-2}^{(n-1)} & \cdots & {}^{\bar{n}}\Sigma_2^{(n-1)} & {}^{\bar{n}}\Sigma_1^{(n-1)} \end{vmatrix} $$

To complete this calculations, we see the following lemma:

Lemma 7.11. *Set*

$$ \eta_k[j] \quad := \begin{vmatrix} {}^{\overline{n-k+2}}E_k[j] & {}^{\overline{n-k+2}}\Sigma_{k-2}^{(k-1)} & \cdots & {}^{\overline{n-k+2}}\Sigma_2^{(k-1)} & {}^{\overline{n-k+2}}\Sigma_1^{(k-1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}E_k[j] & {}^{\bar{n}}\Sigma_{k-2}^{(k-1)} & \cdots & {}^{\bar{n}}\Sigma_2^{(k-1)} & {}^{\bar{n}}\Sigma_1^{(k-1)} \end{vmatrix}, \tag{7.33} $$

$$ \phi_k \quad := \begin{vmatrix} {}^{\overline{n-k+2}}F_k & {}^{\overline{n-k+2}}\Sigma_{k-2}^{(k-1)} & \cdots & {}^{\overline{n-k+2}}\Sigma_2^{(k-1)} & {}^{\overline{n-k+2}}\Sigma_1^{(k-1)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{n}}F_k & {}^{\bar{n}}\Sigma_{k-2}^{(k-1)} & \cdots & {}^{\bar{n}}\Sigma_2^{(k-1)} & {}^{\bar{n}}\Sigma_1^{(k-1)} \end{vmatrix}. \tag{7.34} $$

Then we have

$$ \eta_k[j] = \eta_{k-1}[j] + \frac{{}^{\overline{n-k+2}}E_k[j](c)}{c_{k-2}^{(k-1)}}, \tag{7.35} $$

$$ \phi_k = \phi_{k-1} + \left(\frac{1}{c_{k-2}^{(k-1)}} + \eta_{k-1}[k-1]\right) {}^{\overline{n-k+2}}F_k(c). \tag{7.36} $$

Proof of Lemma7.11 By the formula (7.28) and (7.31), we obtain

$$ {}^{\overline{n-k+2}}E_k[j]({}^{\bar{i}}\Sigma_{k-2}^{(k-1)} - c_{k-3}^{(k-1)} \cdot {}^{\bar{i}}E_k[j]) = \frac{c_{k-3}^{(k-1)}}{c_{k-2}^{(k-1)}} {}^{\overline{n-k+2}}E_k[j] \cdot {}^{\bar{i}}\Sigma_{k-2}^{(k-2)} + c_{k-3}^{(k-1)} \cdot {}^{\bar{i}}E_{k-1}[j]. \tag{7.37} $$

Considering the column transformations as above to the determinant in (7.33) and applying the formula (7.28) and (7.37) repeatedly, we obtain (7.35). Similarly, applying (7.28) and (7.30) to the determinant in (7.34) repeatedly, we obtain (7.36). ◻

From (7.32), we have

$$\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \frac{{}^{\overline{1}}\Sigma^{(n)}_{\overline{n}}}{c^{(n)}_{n-1}} + {}^{\overline{1}}\Sigma^{(n)}_{\overline{n}}\cdot\eta_n[n]+\phi_n. \tag{7.38}$$

By (7.35) and (7.36), we get that

$$\eta_k[j] = \sum_{l=2}^{k} c^{(l-1)^{-1}}_{l-2}\,{}^{\overline{n-l+2}}E_l[j](c), \tag{7.39}$$

$$\phi_k = \phi_2 + \sum_{j=3}^{k} {}^{\overline{n-j+2}}F_j(c)(c^{(j-1)^{-1}}_{j-2} + \eta_{j-1}[j-1]). \tag{7.40}$$

Substituting these to (7.38), we have

$$\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \phi_2 + \sum_{j=3}^{n+1} {}^{\overline{n-j+2}}F_j(c)\cdot(c^{(j-1)^{-1}}_{j-2} + \sum_{l=2}^{j-1} c^{(l-1)^{-1}}_{l-2}\,{}^{\overline{n-l+2}}E_l[j-1](c)), \tag{7.41}$$

where we understand ${}^{\overline{1}}F_{n+1}(c)=1$. Using (7.35) and the explicit form of ${}^{\overline{i}}E_k[j]$ in (7.26), we get the following theorem:

Theorem 7.12. *We have*

$$\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \sum_{(*)} D^{(l_1)}_{l_1}\cdot D^{(l_2-1)}_{l_2}\cdots D^{(l_k-k+1)}_{l_k}\cdot \overline{D}^{(2)}_{u_1}\overline{D}^{(3)}_{u_2}\cdots\overline{D}^{(m+1)}_{u_m}, \tag{7.42}$$

where $D^{(j)}_i = \frac{c^{(j)}_i}{c^{(j)}_{i-1}}$, $\overline{D}^{(j)}_i = \frac{c^{(j-1)}_{i-1}}{c^{(j)}_i}$ *and* $(*)$ *is the conditions:* $k+m=n$, $0\le m<n$, $1\le l_1<l_2<\cdots<l_k\le n$, *and* $1\le u_1<u_2<\cdots<u_m\le n$.

Proof. For $M\in\mathcal{M}^{(p)}_i$ if $i=n-k+2$ and $M_S\neq\emptyset$, then $L=1$ and then $M=M_S=\{k\}$. Thus, it follows from (7.26) and (7.27) that

$$ {}^{\overline{n-k+2}}E_k[j](c) = \sum_{\substack{k\le u_l<u_{k+1}\\ <\cdots<u_j\le n}} \frac{c^{(k-1)}_{u_k-1}c^{(k)}_{u_{k+1}-1}\cdots c^{(j-1)}_{u_j-1}}{c^{(k)}_{u_k}c^{(k+1)}_{u_{k+1}}\cdots c^{(j)}_{u_j}} = \sum_{\substack{k\le u_l<u_{k+1}\\ <\cdots<u_j\le n}} \overline{D}^{(k)}_{u_k}\overline{D}^{(k+1)}_{u_{k+1}}\cdots\overline{D}^{(j)}_{u_j}, \tag{7.43}$$

$$ {}^{\overline{n-k+2}}F_k(c) = \sum_{1\le l_1<l_2<\cdots<l_{n-k+2}\le n} D^{(l_1)}_{l_1}D^{(l_2-1)}_{l_2}\cdots D^{(l_{n-k+2}-(n-k+1))}_{l_{n-k+2}}. \tag{7.44}$$

$$\frac{1}{c^{(l-1)}_{l-2}} = \overline{D}^{(2)}_1\overline{D}^{(3)}_2\cdots\overline{D}^{(l-1)}_{l-2}. \tag{7.45}$$

Then, applying these to (7.41) we get (7.42) ◻

Since we have

$$\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})),$$

where for $c=(c^{(j)}_i)$, we define $\overline{c}=(c^{(n-j+1)^{-1}}_i)$, we get the following:

$$\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{(*)} C^{(1)}_{u_1}C^{(2)}_{u_2}\cdots C^{(m)}_{u_m}\overline{C}^{(l_1-1)}_{l_1}\cdot\overline{C}^{(l_2-2)}_{l_2}\cdots\overline{C}^{(l_k-k)}_{l_k} \tag{7.46}$$

where $C_i^{(j)}=\frac{c_i^{(n-j)}}{c_{i-1}^{(n-j+1)}}$, $\overline{C}_i^{(j)}=\frac{c_{i-1}^{(n-j)}}{c_i^{(n-j)}}$ and $(*)$ is the conditions: $k+m=n$, $0\le m<n$, $1\le l_1<l_2<\cdots<l_k\le n$, and $1\le u_1<u_2<\cdots<u_m\le n$.

This shows (7.3) and then we accomplish the proof of Theorem 7.2. □

7.4. **Correspondence to the monomial realizations.** Let us present the affirmative answer to the conjecture for type C_n in this subsection. First, we see the monomial realization of $B(\Lambda_1)$ associated with the cyclic oder $\cdots(12\cdots n)(12\cdots n)\cdots$, which means that the sign $p_0=(p_{i,j})$ is given by $p_{i,j}=1$ if $i<j$ and $p_{i,j}=0$ if $i>j$. The crystal $B(\Lambda_1)$ is described as follows: We abuse the notation $B(\Lambda_1):=\{v_i,v_{\overline{i}}|1\le i\le n\}$ if there is no confusion. Then the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined as $\tilde{f}_i=f_i$ and $\tilde{e}_i=e_i$ in (3.2) and (3.3). To describe the monomial realizations, we write down the monomials $A_{i,m}$ associated with p_0:

$$A_{i,m}=\begin{cases} c_1^{(m)}c_2^{(m)^{-1}}c_1^{(m+1)}, & \text{for } i=1,\\ c_i^{(m)}c_{i+1}^{(m)^{-1}}c_{i-1}^{(m+1)^{-1}}c_i^{(m+1)}, & \text{for } 1<i\le n-1,\\ c_n^{(m)}c_{n-1}^{(m+1)^{-2}}c_n^{(m+1)} & \text{for } i=n.\end{cases} \tag{7.47}$$

Here the monomial realization of $B(\Lambda_1)$ is described explicitly:

$$B(c_1^{(k)})(\cong B(\Lambda_1))=\left\{\frac{c_j^{(k)}}{c_{j-1}^{(k+1)}}=m_1^{(k)}(v_j),\ \frac{c_{j-1}^{(k+n-j+1)}}{c_j^{(k+n-j+1)}}=m_1^{(k)}(v_{\overline{j}})\mid 1\le j\le n\right\}, \tag{7.48}$$

where $m_i^{(k)}:B(\Lambda_i)\hookrightarrow\mathcal{Y}(p)$ $(u_{\Lambda_i}\mapsto c_i^{(k)})$ is the embedding of crystal as in Sect.5 and we understand $c_0^{(k)}=1$. Now, Theorem 7.1 claims the following:

Theorem 7.13. *We obtain* $\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=m_n^{(n)}(u_{\Lambda_n})=c_n^{(n)}$ *and*

$$\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^{n}m_1^{(k)}(v_j)+\sum_{j=k+1}^{n}m_1^{(k)}(v_{\overline{j}}),\qquad(k=1,2\cdots,n-1) \tag{7.49}$$

$$\Delta_{w_0s_k\Lambda_k,\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^{k}m_1^{(k-n)}(v_{\overline{j}}),\qquad(k=1,2,\cdots,n-1). \tag{7.50}$$

To mention the result for $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$, we define the following set of monomials:

$$\mathbb{B}:=\left\{C_{u_1}^{(1)}C_{u_2}^{(2)}\cdots C_{u_m}^{(m)}\overline{C}_{l_1}^{(l_1-1)}\cdot\overline{C}_{l_2}^{(l_2-2)}\cdots\overline{C}_{l_k}^{(l_k-k)}\ \middle|\ \begin{array}{l}0\le m\le n,\ k+m=n,\\ 1\le l_1<l_2<\cdots<l_k\le n,\\ 1\le u_1<u_2<\cdots<u_m\le n,\\ l_a=u_b\Rightarrow l_a\ge a+b\end{array}\right\} \tag{7.51}$$

where $C_i^{(j)}=\frac{c_i^{(n-j)}}{c_{i-1}^{(n-j+1)}}$, $\overline{C}_i^{(j)}=\frac{c_{i-1}^{(n-j)}}{c_i^{(n-j)}}$ as above. Note that if $k=0$ (resp. $m=0$) for an element in $\mathbb{B}$, then it is just $c_n^{(0)}$ (resp. $\frac{1}{c_n^{(n)}}$). We find the following fact:

Proposition 7.14. *Let* $m_n^{(0)}:B(\Lambda_n)\to\mathcal{Y}(p)$ *be the embedding of crystals such that* $m_n^{(0)}(u_{\Lambda_n})=c_n^{(0)}$. *Then, we obtain* $\mathrm{Im}(m_n^{(0)})=\mathbb{B}$ *and then* $B(c_n^{(0)})=\mathbb{B}$.

Proof. The map $m_n^{(0)}$ is described explicitly:

$$m_n^{(0)}([u_1,\cdots,u_m,\overline{l_k},\overline{l_{k-1}},\cdots,\overline{l_1}])=C_{u_1}^{(1)}C_{u_2}^{(2)}\cdots C_{u_m}^{(m)}\overline{C}_{l_1}^{(l_1-1)}\cdot\overline{C}_{l_2}^{(l_2-2)}\cdots\overline{C}_{l_k}^{(l_k-k)}, \tag{7.52}$$

where $[u_1,\cdots,u_m,\overline{l_k},\overline{l_{k-1}},\cdots,\overline{l_1}]$ is an element in $B(\Lambda_n)$ (see (3.5)), which satisfies

$$u_1<u_2\cdots<u_m\le n<\overline{l_k}<\overline{l_{k-1}}<\cdots<\overline{l_1}.$$

Note that $m_n^{(0)}([1,2,\cdots,n])=c_n^{(0)}$ and $m_n^{(0)}([\overline{n},\overline{n-1},\cdots,\overline{2},\overline{1}])=\frac{1}{c_n^{(n)}}$, which are the highest weight vector and the lowest weight vector in $B(c_n^{(0)})$ respectively. So, it suffices to see the compatibility $\tilde{e}_i\circ m_n^{(0)}=m_n^{(0)}\circ\tilde{e}_i$ and $\tilde{f}_i\circ m_n^{(0)}=m_n^{(0)}\circ\tilde{f}_i$. To see these, we find the parts including ${c_i^{(j)}}^{\pm}$. Indeed, $C_i^{(j)},C_{i+1}^{(j)},\overline{C}_i^{(j)}$ and $\overline{C}_{i+1}^{(j)}$ include $c_i^{(n-j)},{c_i^{(n-j+1)}}^{-1},{c_i^{(n-j)}}^{-1}$ and $c_i^{(n-j)}$ respectively. There can be $16=2^4$ cases according that each of the 4parts exists or not, which correspond to the vectors in $B(\Lambda_n)$ including $i,i+1,\overline{i+1}$ and $\overline{i}$. For example, if we consider $v=[\cdots,i,\cdots,]$, that is, if v includes only i not $i+1,\overline{i+1},\overline{i}$, then we have $m_n^{(0)}(v)$ includes the part $C_i^{(k)}$ where the entry i in v is the k-th entry. In this case, $m_n^{(0)}(v)$ has $c_i^{(n-k)}$ and then $\varphi_i(m_n^{(0)}(v))=1$ and $\varepsilon_i(m_n^{(0)}(v))=0$. Then, applying $\tilde{f}_i$ on $m_n^{(0)}(v)$, we have $\tilde{f}_i(m_n^{(0)}(v))=m_n^{(0)}(v)\cdot A_{i,n-k}^{-1}$. Here we find

$$C_i^{(k)}\cdot A_{i,n-k}^{-1}=\begin{cases} c_1^{(n-k)}\cdot\dfrac{c_2^{(n-k)}}{c_1^{(n-k)}c_1^{(n-k+1)}}=\dfrac{c_2^{(n-k)}}{c_1^{(n-k+1)}}=C_2^{(k)}, & \text{for } i=1,\\ \dfrac{c_i^{(n-k)}}{c_{i-1}^{(n-k+1)}}\cdot\dfrac{c_{i+1}^{(n-k)}c_{i-1}^{(n-k+1)}}{c_i^{(n-k)}c_i^{(n-k+1)}}=\dfrac{c_{i+1}^{(n-k)}}{c_i^{(n-k+1)}}=C_{i+1}^{(k)} & \text{for } 1<i<n,\\ \dfrac{c_n^{(n-k)}}{c_{n-1}^{(n-k+1)}}\cdot\dfrac{{c_{n-1}^{(n-k+1)}}^2}{c_n^{(n-k)}c_n^{(n-k+1)}}=\dfrac{c_{n-1}^{(n-k+1)}}{c_n^{(n-k+1)}}=\overline{C}_n^{(k-1)} & \text{for } i=n.\end{cases}\tag{7.53}$$

These correspond to

$$\tilde{f}_i(v)=\begin{cases}[2,\cdots] & \text{for } i=1,\\ [\cdots,i+1,\cdots] & \text{for } 1<i<n,\\ [\cdots,\overline{n},\cdots] & \text{for } i=n.\end{cases}$$

Then, in this case we obtain $\tilde{f}_i(m_n^{(0)}(v))=m_n^{(0)}(\tilde{f}_i(v))$.

Next, we consider $v=[\cdots,\overline{i},\cdots,]$, that is, if v includes only $\overline{i}$ not $i,i+1,\overline{i+1}$, then we have $m_n^{(0)}(v)$ includes the part $\overline{C}_i^{(k)}$ where the entry $\overline{i}$ in v is the $i-k$-th entry from the bottom. In this case, $m_n^{(0)}(v)$ has ${c_i^{(n-k)}}^{-1}$ and then $\varepsilon_i(m_n^{(0)}(v))=1$. Then, applying $\tilde{e}_i$ on $m_n^{(0)}(v)$, we have $\tilde{e}_i(m_n^{(0)}(v))=m_n^{(0)}(v)\cdot A_{i,n-k-1}$. Here we find

$$\overline{C}_i^{(k)}\cdot A_{i,n-k-1}=\begin{cases} {c_1^{(n-k)}}^{-1}\cdot\dfrac{c_1^{(n-k-1)}c_1^{(n-k)}}{c_2^{(n-k-1)}}=\dfrac{c_1^{(n-k-1)}}{c_2^{(n-k-1)}}=\overline{C}_2^{(k+1)}, & \text{for } i=1,\\ \dfrac{c_{i-1}^{(n-k)}}{c_i^{(n-k)}}\cdot\dfrac{c_i^{(n-k-1)}c_i^{(n-k)}}{c_{i+1}^{(n-k-1)}c_{i-1}^{(n-k)}}=\dfrac{c_i^{(n-k-1)}}{c_{i+1}^{(n-k-1)}}=\overline{C}_{i+1}^{(k+1)} & \text{for } 1<i<n,\\ \dfrac{c_{n-1}^{(n-k)}}{c_n^{(n-k)}}\cdot\dfrac{c_n^{(n-k-1)}c_n^{(n-k)}}{{c_{n-1}^{(n-k)}}^2}=\dfrac{c_n^{(n-k-1)}}{c_{n-1}^{(n-k)}}=C_n^{(k+1)} & \text{for } i=n.\end{cases}\tag{7.54}$$

These correspond to

$$\tilde{e}_i(v)=\begin{cases}[\cdots,\overline{2}] & \text{for } i=1,\\ [\cdots,\overline{i+1},\cdots] & \text{for } 1<i<n,\\ [\cdots,n,\cdots] & \text{for } i=n.\end{cases}$$

Then, in this case we obtain $\tilde{e}_i(m_n^{(0)}(v))=m_n^{(0)}(\tilde{e}_i(v))$. As for other cases, we can discuss similarly and obtain $\tilde{e}_i(m_n^{(0)}(v))=m_n^{(0)}(\tilde{e}_i(v))$ and $\tilde{f}_i(m_n^{(0)}(v))=m_n^{(0)}(\tilde{f}_i(v))$. $\square$

Finally, we shall show

Proposition 7.15. *Let $\widetilde{\mathbb{B}}$ be the set of monomials:*

$$\widetilde{\mathbb{B}} := \left\{ C^{(1)}_{u_1} C^{(2)}_{u_2} \cdots C^{(m)}_{u_m} \overline{C}^{(l_1-1)}_{l_1} \cdot \overline{C}^{(l_2-2)}_{l_2} \cdots \overline{C}^{(l_k-k)}_{l_k} \,\middle|\, \begin{array}{l} 0 \le m \le n,\ k+m=n, \\ 1 \le l_1 < l_2 < \cdots < l_k \le n, \\ 1 \le u_1 < u_2 < \cdots < u_m \le n. \end{array} \right\} \tag{7.55}$$

Then, we get $\mathbb{B} = \widetilde{\mathbb{B}}$.

Here, note that the difference between $\mathbb{B}$ and $\widetilde{\mathbb{B}}$ is the condition "$l_a = u_b \Rightarrow l_a \ge a+b$". Thus, the inclusion $\mathbb{B} \subset \widetilde{\mathbb{B}}$ is trivial.

To show Proposition 7.15, we see the following: Let $\mathcal{B}$ be the set

$$\mathcal{B} := \left\{ [j_1, \cdots, j_n] \,\middle|\, 1 \le j_1 < \cdots < j_k \le \overline{1}. \right\}. \tag{7.56}$$

Then, it follows from (3.5) that $B(\Lambda_n) \subset \mathcal{B}$. We extend the map $m^{(0)}_n$ to the map on $\mathcal{B}$, say also $m^{(0)}_n$. So, for $v = [u_1, \cdots, u_m, \overline{l_k}, \overline{l_{k-1}}, \cdots, \overline{l_1}] \in \mathcal{B}$, we have the monomial

$$m^{(0)}_n([u_1, \cdots, u_m, \overline{l_k}, \overline{l_{k-1}}, \cdots, \overline{l_1}]) = C^{(1)}_{u_1} C^{(2)}_{u_2} \cdots C^{(m)}_{u_m} \overline{C}^{(l_1-1)}_{l_1} \cdot \overline{C}^{(l_2-2)}_{l_2} \cdots \overline{C}^{(l_k-k)}_{l_k}, \tag{7.57}$$

which appears in the summation (7.46) and may not necessarily belong to $\mathbb{B}$. But, indeed, we can show that it belongs to $\mathbb{B}$, which means that the conjecture is positive for this case.

For $v = [u_1, u_2, \cdots, \overline{l}_2, \overline{l}_1] \in \mathcal{B}$, if $u_a = i = l_b$ and $a + b \le i$, then we say the pair $(u_a, \overline{l}_b)$ in v is in *i-configuration* or simply, v is in *i-configuration*. Thus, if $v \in \mathcal{B}$ is in i-configuration for any i, then $v \in B(\Lambda_n)$. For this v, let $(j : k) = j, j+1, \cdots, k-1, k$ (resp. $\overline{(s : t)} = \overline{s}, \overline{s-1}, \cdots, \overline{t}$) be a consecutive subsequence of $u_1, \cdots, u_m$ (resp. $\overline{l}_k, \overline{l_{k-1}}, \cdots, \overline{l_1}$), which is also called a segment of v.

Lemma 7.16. *For $v = [u_1, u_2, \cdots, \overline{l}_2, \overline{l}_1] \in \mathcal{B}$, suppose that there exist a, b such that $u_a = j = l_b$ and v is not in j-configuration.*

(*i*) *For $v = (\cdots (j : m) \cdots \overline{(k : j)} \cdots)$ we have $n > m + k - j$, where $(j : m)$ and $\overline{(k, j)}$ are segments of v.*

(*ii*) *We assume that a is the smallest among the elements satisfying $u_a = l_b = j$ and $a + b > j$ for some b. Then we have $a + b = j + 1$.*

Proof. (i) Since v is not in j-configuration, we have $a + b > j$. If $u_c = m$ and $l_d = k$, $c + d \le n$. We know that $c - a = m - j$ and $d - b = k - j$, which mean that

$$n \ge c + d = (a + m - j) + (b + k - j) > m + k - j.$$

(ii) If there is no c, d such that $u_c = l_d$, $c < a$ and $d < b$. Then we have $(a-1) + (b-1) \le j - 1$ and then $a + b \le j + 1$. Since v is not in j-configuration, we obtain $a + b = j + 1$. If there are c, d such that $c < a$, $d < b$, $u_c = i = l_d$, $i < j$ and v is in i-configuration. We may assume that c, d are the nearest to a, b, that is, there is no pair $(k, \overline{k})$ in v such that $i < k < j$. Indeed, if there is such a pair $(k, \overline{k})$ with $i < k < j$ and v is in k-configuration, then we may replace i with k. Unless v is in k-configuration, it contradicts the minimality of a, b. The last assumption means $(a - c) + (b - d) \le j - i + 1$. Since v is in i-configuration, we have $c + d \le i$. Then we get $a + b \le j + 1$. Thus, by the assumption v is not in j-configuration, we have $a + b > j$ and then we obtain $a + b = j + 1$. $\square$

Definition 7.17. (i) If a pair $(u_a, \overline{l}_b) = (i, \overline{i})$ in v satisfies $a + b = i + 1$, we call such a pair is in *$(+1)$-configuration*.

(ii) We define the set as follows:

$$\mathcal{B}^* := \left\{ v = [u_1, \cdots, u_m, \overline{l}_k, \cdots, \overline{l}_1] \,\middle\|\, v \text{ satisfies the conditions (A1)-(A6) below.} \right\} \tag{7.58}$$

(A1) $0 \le m \le n,\ k + m = n.$

(A2) There exists $x \in \{0,1,\cdots,m\}$ such that $1 \le u_1 < \cdots < u_x$, $u_{x+1} < \cdots < u_m \le n$ and there exists $y \in \{0,1,\cdots,k\}$ such that $1 \le l_1 < \cdots < l_y$, $l_{y+1} < \cdots < l_k \le n$, and x, y satisfy the following (A3)-(A6).

(A3) If there exist $p \in \{1,\cdots,x\}$ and $q \in \{1,\cdots,y\}$ such that $u_p = l_q$, then $p + q \le u_p$.

(A4) If $u_x \ge u_{x+1}$, then there exists $z \in \{1,\cdots,y\}$ such that $l_z = u_x$ and $x + z \le u_x$, and there exists $w \in \{1,2,\cdots,z\}$ such that $l_w = u_{x+1}$, $\{l_z, l_{z+1}, \cdots, l_w\}$ is consecutive and $(x+1) + w = u_{x+1} + 1$, namely, the pair $(u_{x+1}, \bar{l}_w)$ is in (+1)-configuration.

(A5) If $l_y \ge l_{y+1}$, then there exists $z' \in \{1,\cdots,x\}$ such that $u_{z'} = l_y$ and $y + z' \le l_y$, and there exists $w' \in \{1,2,\cdots,z'\}$ such that $u_{w'} = l_{y+1}$, $\{u_{z'}, u_{z'+1}, \cdots, u_{w'}\}$ is consecutive and $w' + (y+1) = l_{y+1} + 1$, namely, the pair $(u_{w'}, \bar{l}_{y+1})$ is in (+1)-configuration.

(A6) If $u_x \ge u_{x+1}$ and $l_y \ge l_{y+1}$, then $u_x = l_y$.

By the definition, it is evident $\mathcal{B} \subset \mathcal{B}^*$. For an element $v = [u_1, \cdots, u_m, \bar{l}_k, \cdots, \bar{l}_1] \in \mathcal{B}^*$, we define its level $l(v)$ as follows:
(7.59)

$$l(v) = \begin{cases} \min(a+b-1 | u_a = l_b \text{ and } a+b = u_a+1) & \text{if there exist } a, b \text{ such that } u_a = l_b, a+b = u_a+1, \\ n & \text{otherwise.} \end{cases}$$

The following lemma is obtained from Lemma 7.16 and the definition of the level.

Lemma 7.18. (*i*) *Assume that for $v \in \mathcal{B}^*$ there exist x (resp. y) such that $u_x \ge u_{x+1}$ (resp. $l_y \ge l_{y+1}$) and there is no y (resp. x) such that $l_y \ge l_{y+1}$ (resp. $u_x \ge u_{x+1}$). Then we have $l(v) = u_{x+1}$ (resp. $l(v) = l_{y+1}$).*

(*ii*) *Assume that for $v \in \mathcal{B}^*$ there exist x, y such that $u_x \ge u_{x+1}$ and $l_y \ge l_{y+1}$. Then we have $l(v) = \min(u_{x+1}, l_{y+1})$.*

(*iii*) *For an element v in $\mathcal{B}^*$, $v \in B(\Lambda_n)$ if and only if $l(v) = n$.*

Proof. The statements (i) and (ii) are evident from the definition of $\mathcal{B}^*$. Let us show (iii). Write $v = [u_1, \cdots, u_k, \bar{l}_m, \cdots, \bar{l}_1]$. If $v \in B(\Lambda_n)$, then any $(i, \bar{i})$-pair in v is in i-configuration, which means that $l(v) = n$. Conversely, assume that $v \notin B(\Lambda_n)$. Consider the case that for v there exists x as in the condition (A4) above. In this case, there exists z such that $u_{x+1} = l_z$ and $(x+1)+z = u_{x+1}+1$, which implies $l(v) \le u_{x+1} < n$. The case that there exists y satisfying the condition (A5) is treated by the similar way. Now, assume that there are no x, y satisfying the conditions (A4) and (A5) respectively. In this case, v is an element in $\mathcal{B}$. It follows from Lemma 7.16 that there are a, b such that $u_a = l_b$ and $a + b = u_a + 1$. This shows that $l(v) \le u_a < n$ since $(n, \bar{n})$-pair is always in n-configuration. □

Definition 7.19. Define the transformation τ_i $(i = 1, 2, \cdots, n-1)$ on $\mathcal{B}^*$ as follows:

(i) For $v = [u_1, \cdots, u_m, \overline{l_{m'}}, \overline{l_{m'-1}}, \cdots, \overline{l_1}]$, find i which is the smallest such that $u_a = i = l_b$ for some a, b and $a + b = i + 1$. If it does not exist, then τ_i is nothing but the identity. Indeed, In this case by Lemma 7.18 $v \in B(\Lambda_n)$. Consider the case that such a exists. Then let $(i : j)$ and $\overline{(k : i)}$ be the segments in v including $i, \bar{i}$ and j, k are the largest ones.

(ii) Suppose there are no x, y satisfying the condition (A4) and (A5) respectively, that is, $v \in \mathcal{B}$. Let $(j_1 : j_2)$ (resp. $\overline{(k_2 : k_1)}$) be the right-next (resp. left-next)segments to $(i : j)$ (resp. $\overline{(k : i)}$), that is, $v = (\cdots (i : j)(j_1 : j_2) \cdots \overline{(k_2 : k_1)}\ \overline{(k : i)} \cdots)$. Let

$$\tau_i : v \mapsto (\cdots (k+1 : k+j-i+1)(j_1 : j_2) \cdots \overline{(k_2 : k_1)}\ \overline{(k+j-i+1 : j+1)} \cdots).$$

If $j \neq i$, then τ_j is the identity.

(iii) Suppose that there is x (resp. y)satisfying (A4) (resp. (A5)) and no y (resp. x)satisfying (A5) (resp. (A4)). In this case, it follows from the above argument that set $i = u_{x+1}$ (resp. $i = l_{y+1}$) and then τ_i is defined as same as the previous one and τ_j is identity for $j \neq i$.

(iv) Suppose that there are x and y satisfying (A4) and (A5) respectively and w and w' are as in (A4) and (A5) respectively, namely, $u_{x+1} = l_z$ and $x + w = u_a$, and $u_{w'} = l_{y+1}$ and $w' + y = l_{y+1}$. If $u_{x+1} \neq l_{y+1}$, then set $i = \min(u_{x+1}, l_{y+1})$ and τ_i is defined as the previous one. If $u_{x+1} = l_{y+1}(= i)$, then τ_i is the composition of the previous $\tau_{u_{x+1}}$ and $\tau_{l_{y+1}}$, that is, $\tau_i := \tau_{l_{y+1}} \circ \tau_{u_{x+1}}$. If $j \neq i$, set $\tau_j = \mathrm{id}$.

We obtain the following lemma:

Lemma 7.20. (*i*) *The transformation τ_i defined above is well-defined, that is, $\tau_i(\mathcal{B}^*) \subset \mathcal{B}^*$ for any i.*

(*ii*) *If $\tau_i \neq \mathrm{id}$, then we have $l(\tau_i(v)) > l(v)$ for any $v \in \mathcal{B}^*$.*

Proof. First let us see the case (ii) in Definition 7.19. In this case, we have

$$\tau_i(v) = (\cdots(k+1 : k+j-i+1)(j_1 : j_2)\cdots\overline{(k_2 : k_1)}\ \overline{(k+j-i+1 : j+1)}\cdots).$$

In this formula, if $k + j - i + 1 < \min(j_1, k_1)$, $\tau_i(v) \in \mathcal{B} \subset \mathcal{B}^*$. We can see that the level of $\tau_i(v)$ is equal to j_1 and it is greater than i, which implies $l(\tau_i(v)) > l(v)$. Consider the case $j_1 \leq k+j-i+1$, which corresponds to the condition (A4) in Definition 7.17. By the definition of the segment we know that $j_1 > j + 1$. Thus, we have $\overline{j_1} \in \overline{(k + j - i + 1 : j + 1)}$, which means there is the pair $(j_1, \overline{j_1})$'s in $\tau_i(v)$. Let us see that this pair is in $(+1)$-configuration. Set $\tau_i(v) := [u'_1, \cdots, u'_k, \overline{l}'_m, \cdots, \overline{l}'_1]$. And let $u'_a = k+1$, $u'_b = j_1$, $l'_c = \overline{j+1}$ and $l'_d = \overline{j_1}$. Thus, we have $b-a = (k+j-i+1)-(k+1)+1 = j-i+1$ and $d - c = j_1 - (j + 1)$. Here note that $u_a = i$ and $l_c = \overline{i}$ in v, which means $a + c = i + 1$. Thus, we get

$$b + d = a + (j - i + 1) + c + (j_1 - j - 1) = a + c + j_1 - i = j_1 + 1.$$

We can easily see that any pair $(i, \overline{i})$ in the parts $(u'_1, \cdots, k + j - i + 1)$ and $(\overline{k + j - i + 1}, \cdots, \overline{l}'_c)$ is in i-configuration. Then, $\tau_i(v)$ is in $\mathcal{B}^*$. As for the level of $\tau_i(v)$, we can show as the previous case. The case $k_1 \leq k + j - i + 1$ is shown similarly, which corresponds to the condition (A5) in Definition 7.17.

Next, let us see (iii) in Definition 7.19. Suppose that there is x such that $u_x \geq u_{x+1}$ and there is no y such that $l_y \geq l_{y+1}$. Let w be the number such that $u_{x+1} = l_w$ and $x + w = u_{x+1}$. Set $i := u_x \geq u_{x+1} =: j = l_w$ and let $(j : A)$ (resp. $\overline{(B : j)}$) be the segment including $j = u_{x+1}$ (resp. $\overline{j} = \overline{l}_w$) and $A = u_z$ (resp. $B = l_{z'}$), that is,

$$v = [\cdots i, (j : A), A', \cdots, \overline{B}', \overline{(B : j)} \cdots]$$

Note that there is $\overline{i}$ between $\overline{B}$ and $\overline{j}$. Then, we obtain

$$\tau_j(v) = [\cdots i, (B + 1 : A + B - j + 1), A', \cdots, \overline{B}', \overline{(A + B - j + 1 : A + 1)} \cdots] \tag{7.60}$$

Here $i < B + 1$ and there are the following two cases:
(1) $A + B - j + 1 < A', B'$. (2) $A + B - j + 1 \geq \min(A', B')$.
In the first case, we do not need to consider the conditions (A4)–(A6) and then $\tau_j(v) \in \mathcal{B} \subset \mathcal{B}^*$. Let us see the case (2). By the similar argument to the above cases (ii) and (iii), we know that $\tau_j(v) \in \mathcal{B}^*$ and $l(\tau_j(v)) \geq \min(A', B') > j \geq l(v)$. We can also check the case that there is y such that $l_y \geq l_{y+1}$ and there is no x such that $u_x \geq u_{x+1}$.

Finally, let us see (iv) in Definition 7.19. Let x, y be numbers satisfying $u_x \geq u_{x+1}$ and $l_y \geq l_{y+1}$ and set $i := u_x = l_y$, $j := u_{x+1}$, $k := l_{y+1}$. Note that $i \geq j, k$. By the condition (A6), there exist r, s such that $u_r = k$, $l_s = j$, $(x+1)+s = j+1$ and $(y+1)+r = k+1$. Note that $l(v) = \min(j, k)$. Now, v is in the form:

$$v = [\cdots (k : i), (j : A), B, \cdots \overline{C}, \overline{(D : k)}, \overline{(i : j)} \cdots]$$

where $(j : A)$ (resp. $\overline{(D : k)}$) is the segment including j (resp. $\overline{k}$), $A < B + 1$ and $C + 1 > D$. We consider the following three cases: (1) $j < k$. (2) $j > k$. (3) $j = k$.

The cases (1) and (2) are treated in the similar manner to the above cases. Thus, we see the case (3). In this case, τ_j is defined as the last one in Definition 7.19 (iv). So, we have

$$
\begin{aligned}
\text{(7.61)}\quad \tau_j(v) &= \tau_{l_{y+1}} \circ \tau_{u_{x+1}}(v) \\
&= \tau_{l_{y+1}}([\cdots,(j:i)(i+1:A+i-j+1),(B:E),F,\cdots,\overline{C},\overline{(D:j)},\overline{(A+i-j+1:j+1)},\cdots]) \\
&= \begin{cases} [\cdots(D+1:G),(B:E),F,\cdots,\overline{C},\overline{(G:A+1)},\cdots] & \text{if } A+i-j\neq B, \\ [\cdots(D+1:H),F,\cdots,\overline{C},\overline{(H:E+1)},\cdots] & \text{if } A+i-j=B, \end{cases}
\end{aligned}
$$

where $G=A+i-2j+D+1$ and $H=E+D-j+1$. Here, we can easily see that $\tau_j(v)$ is in $\mathcal{B}^*$ by the similar way to the previous cases. By the formula (7.61) we also know that

$$l(v)\leq j(=u_{x+1})<\min(B,C,F)\leq l(\tau_j(v)).$$

Now, we completed proving the lemma 7.20. □

Example 7.21. *For $n=10$ and $v=[123566\overline{6}\,\overline{5}\,\overline{3}\,\overline{2}\,\overline{1}]\in\mathcal{B}$, we have*

$$[123566\overline{6}\,\overline{5}\,\overline{3}\,\overline{2}\,\overline{1}]\xrightarrow{\tau_1}[456566\overline{6}\,\overline{5}\,\overline{6}\,\overline{5}\,\overline{4}]\xrightarrow{\tau_5}[456786\overline{6}\,\overline{5}\,\overline{8}\,\overline{7}\,\overline{4}]\xrightarrow{\tau_5}[478788\overline{8}\,\overline{7}\,\overline{8}\,\overline{7}\,\overline{4}]\xrightarrow{\tau_7}[4789\,10\,\overline{10}\,\overline{9}\,\overline{8}\,\overline{7}\,\overline{4}]\in B(\Lambda_n).$$

Proposition 7.22. *For any element $v\in\mathcal{B}^*$, there exists a sequence of indices $i_1,\cdots,i_k$ such that $\tau_{i_1}\circ\cdots\circ\tau_{i_k}(v)\in B(\Lambda_n)$.*

Proof. It follows from Lemma 7.20(ii) that for any $v\in\mathcal{B}^*\setminus B(\Lambda_n)$, there exists j such that $l(v)<l(\tau_j(v))$. This fact means that there exists $i_1,\cdots,i_k$ such that

$$l(\tau_{i_1}\circ\cdots\circ\tau_{i_k}(v))=n,$$

which is equivalent to $\tau_{i_1}\circ\cdots\circ\tau_{i_k}(v)\in B(\Lambda_n)$ by Lemma 7.18. □

Next lemma is the key for the relations to the monomial realization.

Lemma 7.23. *For any $v\in\mathcal{B}^*$ and any $i\in I$, we have $m_n^{(0)}(\tau_i(v))=m_n^{(0)}(v)$.*

Proof. If $\tau_i=\mathrm{id}$, there is nothing to show. So, we consider an element in $\mathcal{B}^*\setminus B(\Lambda_n)$. For $v=[u_1,\cdots,u_m,\overline{l}_m,\cdots,\overline{l}_1]\in\mathcal{B}^*\setminus B(\Lambda_n)$, set $u_a=i=l_b$, $a+b=i+1$ and let $(i:j)$ (resp. $(\overline{(k:i)})$) be a segment including i (resp. $\overline{i}$), that is, $v=[\cdots,(i:j)\cdots,\overline{(k:i)}\cdots]$. Let

$$P:=(C_i^{(a)}C_{i+1}^{(a+1)}\cdots C_j^{(j-i+a)})\cdot(\overline{C}_k^{(b-i,)}\overline{C}_{k-1}^{(b-i)}\cdots\overline{C}_i^{(i-b)})=\frac{c_j^{(n-j+i-a)}}{c_k^{(n-a+1)}}$$

be the part of the monomial $m_n^{(0)}(v)$ related to the segments $(i:j)$ and $\overline{(k:i)}$. Note that for the last equality we use $a+b=i+1$. For the element

$$\tau_i(v)=[\cdots(k+1:k+j-i+1)\cdots\overline{(k+j-i+1:j+1)}\cdots],$$

we also get the part of $m_n^{(0)}(\tau_i(v))$:

$$Q=(C_{k+1}^{(a)}C_{k+2}^{(a+1)}\cdots C_{k+j-i+1}^{(j-i+a)})\cdot(\overline{C}_{k+j-i+1}^{(j-b+1,)}\overline{C}_{k+j-i}^{(j-b+1)}\cdots\overline{C}_{j+1}^{(j-b+1)})=\frac{c_j^{(n-j+i-a)})}{c_k^{(n-a+1)}},$$

which shows $P=Q$ and then $m_n^{(0)}(v)=m_n^{(0)}(\tau_i(v))$. □

Here, let us see that

Lemma 7.24. *For any $v\in\mathcal{B}^*$ there exists a unique element $v'\in B(\Lambda_n)$ which is obtained by applying τ_j's to v.*

Proof. For $v \in \mathcal{B}^* \setminus B(\Lambda_n)$, suppose that there exist $i_1, \cdots, i_r$ and $j_1, \cdots, j_s$ such that

$$v_1 := \tau_{i_1} \cdots \tau_{i_r}(v) \neq \tau_{j_1} \cdots \tau_{j_s}(v) =: v_2,$$

and $v_1, v_2 \in B(\Lambda_n)$. By Lemma 7.23 we know that

$$m_n^{(0)}(v_1) = m_n^{(0)}(v) = m_n^{(0)}(v_2). \tag{7.62}$$

In the meanwhile, the restricted map $m_n^{(0)}{}_{|B(\Lambda_n)}$ is bijective. Thus, it follows from (7.62) that $v_1 = v_2$, which contradicts the assumption $v_1 \neq v_2$. □

Indeed, in the example above, the second step and the third step can be exchanged. But, the result turns out to be the same.

For $v \in \mathcal{B}^*$, we showed that there exists a unique element $v' \in B(\Lambda_n)$ obtained by applying τ_i's. Let us denote v' by $\mathrm{Rect}(v)$ and call it the *rectification* of v. Indeed, we know that $\mathrm{Rect}(\mathcal{B}^*) = \mathrm{Rect}(\mathcal{B}) = B(\Lambda_n)$.

Proof of Proposition 7.15. We have $m_n^{(0)}(\mathcal{B}) = \widetilde{\mathbb{B}}$ and we know that $\mathrm{Rect}(\mathcal{B}) = B(\Lambda_n)$ and $m_n^{(0)}(\mathrm{Rect}(v)) = m_n^{(0)}(v)$ for any $v \in \mathcal{B}$. Therefore, we obtain

$$\widetilde{\mathbb{B}} = m_n^{(0)}(\mathcal{B}) = m_n^{(0)}(\mathrm{Rect}(\mathcal{B})) = m_n^{(0)}(B(\Lambda_n)) = \mathbb{B}.$$ □

Hence, owing to Theorem 7.2 (ii) and Proposition 7.15 one gets the following result:

Theorem 7.25. *For $b \in B(\Lambda_n)$ let $n(b)$ be the multiplicity of b defined as $n(b) := \sharp\{v \in \mathcal{B} | \mathrm{Rect}(v) = b\}$. Then, we have*

$$\Delta_{w_0 s_n \Lambda_n, \Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{b \in B(\Lambda_n) \setminus \{u_{\Lambda_n}\}} n(b) \cdot m_n^{(0)}(b), \tag{7.63}$$

where $u_{\Lambda_n} = [1, 2, \cdots, n-1, n]$ is the highest weight vector.

Thus, we know that Conjecture 6.8 is affirmative for type C_n and the sequence $\mathbf{i_0} = (12 \cdots n)^n$.

8. Explicit form of $f_B(t\Theta^-_{\mathbf{i_0}}(c))$ for B_n

8.1. **Main theorems.** In case of type B_n fix the sequence $\mathbf{i_0} = (12 \cdots n)^n$.

Theorem 8.1. *For $k = 1, \cdots, n$ and $c = (c_j^{(i)})_{1 \leq, i,j \leq n} = (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(n)}, c_n^{(n)}) \in (\mathbb{C}^\times)^{n^2}$, we have*

$$\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))$$
$$= c_1^{(k)} + \frac{c_2^{(k)}}{c_1^{(k+1)}} + \cdots + \frac{c_{n-1}^{(k)}}{c_{n-2}^{(k+1)}} + \frac{c_n^{(k)}{}^2}{c_{n-1}^{(k+1)}} + 2\frac{c_n^{(k)}}{c_n^{(k+1)}} + \frac{c_{n-1}^{(k+1)}}{c_n^{(k+1)}{}^2} + \frac{c_{n-2}^{(k+2)}}{c_{n-1}^{(k+2)}} + \cdots + \frac{c_k^{(n)}}{c_{k+1}^{(n)}},$$

where note that $\Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = c_n^{(n)}$.

Theorem 8.2. *Let k be an index running over $\{1, 2, \cdots, n-1\}$ and c be as in the previous theorem. Then we have*

$$\Delta_{w_0 s_k \Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \frac{1}{c_1^{(k)}} + \sum_{j=1}^{k-1} \frac{c_{k-j}^{(j)}}{c_{k-j+1}^{(j)}}. \tag{8.1}$$

The case $k = n$ will be presented in 8.5. We shall prove the above theorems in 8.2 and 8.3.

8.2. **Proof of Theorem 8.1.** Considering similarly to type C_n as in 7.2, we can write

$$\boldsymbol{x}_i(c) := \alpha_i^\vee(c^{-1})x_i(c) = \begin{cases} c^{-h_i}(1+c\cdot e_i) & i\neq n, \\ c^{-h_n}(1+c\cdot e_n+\frac{c^2}{2}e_n^2) & i=n, \end{cases}$$

$$\boldsymbol{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) = \begin{cases} (1+c\cdot f_i)c^{-h_i} & i\neq n, \\ (1+c\cdot f_n+\frac{c^2}{2}f_n^2)c^{-h_n} & i=n, \end{cases}$$

since $f_i^2=e_i^2=0$ ($i\neq n$) and $e_n^3=f_n^3=0$ on the vector representation $V(\Lambda_1)$. We also have $\omega(\boldsymbol{y}_i(a))=\alpha_i^\vee(a^{-1})x_i(a)=\boldsymbol{x}_i(a)$ and define ${}^{\overline{i}}\Xi_j^{(p)}={}^{\overline{i}}\Xi_j^{(p)}(c^{(p)})$ and ${}^{i}\Xi_j^{(p)}={}^{i}\Xi_j^{(p)}(c^{(p)})$ for $p,j\in I$ by

$$X^{(p)}X^{(p-1)}\cdots X^{(1)}v_i=\sum_{j=0}^n {}^{i}\Xi_j^{(p)}v_j+\sum_{j=1}^n {}^{i}\Xi_{\overline{j}}^{(p)}v_{\overline{j}}\in V(\Lambda_1)\quad (i=0,1,\cdots,n),$$

$$X^{(p)}X^{(p-1)}\cdots X^{(1)}v_{\overline{i}}=\sum_{j=0}^n {}^{\overline{i}}\Xi_j^{(p)}v_j+\sum_{j=1}^n {}^{\overline{i}}\Xi_{\overline{j}}^{(p)}v_{\overline{j}}\in V(\Lambda_1)\quad (i=1,2,\cdots,n),$$

where $c^{(p)}=(c_1^{(1)},c_2^{(1)},\cdots,c_{n-1}^{(p)},c_n^{(p)})$ and $X^{(p)}=\boldsymbol{x}_n(c_n^{(p)})\boldsymbol{x}_{n-1}(c_{n-1}^{(p)})\cdots\boldsymbol{x}_1(c_1^{(p)})$.

By (6.3) and $\omega(\Theta_{\mathbf{i}_0}(c))=X^{(n)}\cdots X^{(1)}$, same as (7.4) we have

$$\Delta_{w_0\Lambda_i,s_i\Lambda_i}(\Theta_{\mathbf{i}}(c))=\langle\overline{s}_i\cdot u_{\Lambda_i}\,,\,X^{(n)}\cdots X^{(1)}v_{\Lambda_i}\rangle.$$

To describe ${}^{\overline{i}}\Xi_j^{(p)}$ explicitly let us use the *segments* as in Sect.7.

For $m\in M=M_1\sqcup\cdots\sqcup M_S\in\mathcal{M}_k^{(p)}$, define $n(m):=n-j+1$ if $m\in M_j$. For $M=M_1\sqcup\cdots\sqcup M_S\in\mathcal{M}_k^{(p)}$, write $M_1=\{2,3,\cdots,a\}$. For $i-1\leq i_2\leq i_3\leq\cdots\leq i_a\leq n$ and for $1\leq b<c\leq a$ and $i\leq j_2\leq\cdots\leq j_b\leq n$, define the monomials in $(c_j^{(i)})_{1\leq i,j\leq n}$ by

$$C^M_{i_2,i_3,\cdots,i_a}:=\frac{\left(c^{(1+\epsilon_{i_2})}_{i_2+1-2\epsilon_{i_2}}\right)^{1+\epsilon_{i_2+1}}\cdots\left(c^{(a+\epsilon_{i_a}-1)}_{i_a+1-2\epsilon_{i_a}}\right)^{1+\epsilon_{i_a+1}}}{{c^{(2)}_{i_2}}^{1+\epsilon_{i_2}}\cdots {c^{(a)}_{i_a}}^{1+\epsilon_{i_a}}},\qquad D^M:=\prod_{m\in M\setminus M_1}\frac{c^{(m)}_{n(m)-1}}{c^{(m)}_{n(m)}},\tag{8.2}$$

$$\widetilde{C}^M_{j_2,j_3,\cdots,j_b;b,c}:=\frac{{c^{(1)}_{j_2}}^{1+\epsilon_{j_2}}\cdots {c^{(b-1)}_{j_b}}^{1+\epsilon_{j_b}}c_n^{(b)}c_{n-1}^{(c+1)}\cdots c_{n-1}^{(a)}}{c^{(2)}_{j_2-1}\cdots c^{(b)}_{j_b-1}c_n^{(c)}{c_n^{(c+1)}}^2\cdots {c_n^{(a)}}^2}.\tag{8.3}$$

where $\epsilon_i=\delta_{i,n}$ and $C^M_{i_2,i_3,\cdots,i_a}=1$ (resp. $D^M=1$) if $M_1=\emptyset$ (resp. $M\setminus M_1=\emptyset$). Note that in (8.3) if $b=1$ (resp. $c=a$), then ${c^{(1)}_{j_2}}^{1+\epsilon_{j_2}}\cdots {c^{(b-1)}_{j_b}}^{1+\epsilon_{j_b}}=c^{(2)}_{j_2-1}\cdots c^{(b)}_{j_b-1}=1$ (resp. $c_{n-1}^{(c+1)}\cdots c_{n-1}^{(a)}={c_n^{(c+1)}}^2\cdots {c_n^{(a)}}^2=1$).

Proposition 8.3. *In the setting above, we have*

$${}^{\overline{i}}\Xi^{(p)}_{\overline{k}}={c^{(1)}_{i-1}}^{-1}\sum_{i\leq i_2\leq\cdots\leq i_p\leq k}(c^{(2)}_{i_2-1}\cdots c^{(p)}_{i_p-1})^{-1}({c^{(1)}_{i_2}}^{1+\epsilon_{i_2}}{c^{(2)}_{i_3}}^{1+\epsilon_{i_3}}\cdots {c^{(p-1)}_{i_p}}^{1+\epsilon_{i_p}}{c^{(p)}_{k}}^{1+\epsilon_{k}}).\tag{8.4}$$

$${}^{\overline{i}}\Xi^{(p)}_{0}={c^{(1)}_{i-1}}^{-1}\sum_{q=1}^{p}\sum_{i\leq i_2\leq\cdots\leq i_q\leq n}(c^{(2)}_{i_2-1}\cdots c^{(q)}_{i_q-1})^{-1}({c^{(1)}_{i_2}}^{1+\epsilon_{i_2}}{c^{(2)}_{i_3}}^{1+\epsilon_{i_3}}\cdots {c^{(q-1)}_{i_q}}^{1+\epsilon_{i_q}}c^{(q)}_{n}),\tag{8.5}$$

$${}^{\overline{i}}\Xi^{(p)}_{k}={c^{(1)}_{i-1}}^{-1}\left(\sum_{(A)}C^M_{i_2,i_3,\cdots,i_a}\cdot D^M+2\sum_{(B)}\widetilde{C}^M_{j_2,\cdots,j_b;b,c}\cdot D^M\right)\tag{8.6}$$

where $c_0^{(1)}=1$ and the conditions (A) and (B) are as follows:

(A) $i-1\le i_2\le\cdots\le i_a\le n$, $M=M_1\sqcup\cdots\sqcup M_S\in\mathcal{M}_k^{(p)}$, $M_1=\{2,\cdots,a\}$.

(B) $i\le j_2\le\cdots\le j_b\le n$, $M=M_1\sqcup\cdots\sqcup M_S\in\mathcal{M}_k^{(p)}$, $M_1=\{2,\cdots,a\}\ne\emptyset$, $1\le b<c\le a$.

Proof. Set $\mathcal{X}:=\boldsymbol{x}_n(c_n)\cdots\boldsymbol{x}_1(c_1)$. Calculating directly we have the formula:

$$\mathcal{X}v_i=\begin{cases}c_1^{-1}v_1 & \text{if } i=1,\\ c_{i-1}c_i^{-1}v_i+v_{i-1} & \text{if } i=2,\cdots,n,\end{cases}\tag{8.7}$$

$$\mathcal{X}v_0=v_0+2c_n^{-1}v_n,\tag{8.8}$$

$$\mathcal{X}v_{\overline{i}}=c_{i-1}^{-1}(c_iv_{\overline{i}}+c_{i+1}v_{\overline{i+1}}+\cdots+c_n^2v_{\overline{n}}+c_nv_0+v_n),\tag{8.9}$$

where we understand $c_0=1$. Using these, we get for $i=1,2,\cdots,n$,

$$^{\overline{i}}\Xi_{\overline{k}}^{(p)}=\sum_{j=i}^{k}{}^{\overline{i}}\Xi_{\overline{j}}^{(p-1)}\frac{{c_k^{(p)}}^{1+\epsilon_k}}{c_{j-1}^{(p)}},\quad (k=1,\cdots,n),\tag{8.10}$$

$$^{\overline{i}}\Xi_{k}^{(p)}={}^{\overline{i}}\Xi_{k+1}^{(p-1)}+{}^{\overline{i}}\Xi_{k}^{(p-1)}\frac{c_{k-1}^{(p)}}{c_k^{(p)}},\quad (k=1\cdots,n-1),\tag{8.11}$$

$$^{\overline{i}}\Xi_{0}^{(p)}=\sum_{j=i}^{n}{}^{\overline{i}}\Xi_{\overline{j}}^{(p-1)}\frac{c_n^{(p)}}{c_{j-1}^{(p)}}+{}^{\overline{i}}\Xi_{0}^{(p-1)},\tag{8.12}$$

$$^{\overline{i}}\Xi_{n}^{(p)}=\sum_{j=i}^{n}{}^{\overline{i}}\Xi_{\overline{j}}^{(p-1)}{c_{j-1}^{(p)}}^{-1}+2\,{}^{\overline{i}}\Xi_{0}^{(p-1)}{c_n^{(p)}}^{-1}+{}^{\overline{i}}\Xi_{n}^{(p-1)}\frac{c_{n-1}^{(p)}}{{c_n^{(p)}}^2}.\tag{8.13}$$

Indeed, the formulae (8.4) and (8.5) are easily shown by the induction on p using the formulae (8.10) and (8.12).

To obtain (8.6) we see the segments of elements in $\mathcal{M}_k^{(p)}$, $\mathcal{M}_{k+1}^{(p-1)}$ and $\mathcal{M}_k^{(p-1)}$ as the case C_n and apply the recursions (8.11) and (8.13) to the induction hypothesis. Arguing similarly to the previous case, we obtain the desired results. □

Thus, for example, we have

$$\Delta_{w_0\Lambda_1,s_1\Lambda_1}={}^{\overline{1}}\Xi_2^{(n)}=\sum_{j=1}^{n-1}\frac{c_j^{(1)}}{c_{j-1}^{(2)}}+\frac{{c_n^{(1)}}^2}{c_{n-1}^{(2)}}+2\frac{c_n^{(1)}}{c_n^{(2)}}+\frac{c_{n-1}^{(2)}}{{c_n^{(2)}}^2}+\sum_{j=3}^{n}\frac{c_{n-j+1}^{(j)}}{c_{n-j+2}^{(j)}}.$$

The following is the same as Lemma 7.6.

Lemma 8.4. *For $k=1,\cdots,n-1$ we define the matrix W_k by*

$$W_k:=\begin{pmatrix}{}^{\overline{1}}\Xi_{k+1}^{(n)} & {}^{\overline{1}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{1}}\Xi_{2}^{(n)} & {}^{\overline{1}}\Xi_{1}^{(n)}\\ {}^{\overline{2}}\Xi_{k+1}^{(n)} & {}^{\overline{2}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{2}}\Xi_{2}^{(n)} & {}^{\overline{2}}\Xi_{1}^{(n)}\\ \vdots & \vdots & \cdots & \vdots & \vdots\\ {}^{\overline{k}}\Xi_{k+1}^{(n)} & {}^{\overline{k}}\Xi_{k-1}^{(n)} & \cdots & {}^{\overline{k}}\Xi_{2}^{(n)} & {}^{\overline{k}}\Xi_{1}^{(n)}\end{pmatrix}.\tag{8.14}$$

Then we have $\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\det W_k$.

The last column of the matrix W_k is just

$$^t({}^{\overline{1}}\Xi_1^{(n)},\cdots,{}^{\overline{k}}\Xi_1^{(n)})={}^t(1,{c_1^{(1)}}^{-1},{c_2^{(1)}}^{-1},\cdots,{c_{k-1}^{(1)}}^{-1}).$$

Then, applying the elementary transformations on W_k by (i-th row) $-\frac{c_i^{(1)}}{c_{i-1}^{(1)}}\times$($i+1$-th row) for $i=1,\cdots,k-1$, in the transformed matrix $\tilde{W}_k$ its (i,j)-entry is as follows:

Lemma 8.5. *For* $c=(c^{(l)}_k)_{1\leq,k,l\leq n}$ *we set* $c^{(l)}:=(c^{(l)}_1,\cdots,c^{(l)}_n)$ *and* $c^{[a:b]}:=(c^{(a)},c^{(a+1)},\cdots,c^{(b)})$ $(a\leq b)$*. For* $i=1,\cdots,k-1$ *the* (i,j)*-entry* $(\tilde{W}_k)_{i,j}$ *is:*

$$(\tilde{W}_k)_{i,j}=\begin{cases}{}^{\overline{i}}\Xi^{(n-1)}_{k+1}(c^{[2:n]}) & \text{if } j=1,\\ {}^{\overline{i}}\Xi^{(n-1)}_{k-j+1}(c^{[2:n]}) & \text{if } 1<j<k,\\ 0 & \text{if } j=k,\end{cases} \tag{8.15}$$

Note that by definition we have $c=c^{[1:n]}$.

Proof. The proof is similar to the one for Lemma 7.7. Indeed, for type B_n we also get the same formula as (7.15):

$${}^{\overline{i}}\Xi^{(n)}_j(c)-\frac{c^{(1)}_i}{c^{(1)}_{i-1}}\,{}^{\overline{i+1}}\Xi^{(n)}_j(c)=\frac{c^{(1)}_i}{c^{(1)}_{i-1}}\,{}^{\overline{i}}\Xi^{(n-1)}_j(c^{[2,n]}).$$

Then, this shows the lemma. □

Applying the above elementary transformations to the matrix W_k, we find

$$\det W_k={}^{\overline{1}}\Xi^{(n-k+1)}_{k+1}(c^{[k:n]}).$$

Thus, it follows from (8.6) that

$$\begin{aligned}\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i}}(c))&={}^{\overline{1}}\Xi^{(n-k+1)}_{k+1}(c^{[k:n]})\\ &=c^{(k)}_1+\frac{c^{(k)}_2}{c^{(k+1)}_1}+\cdots+\frac{c^{(k)}_{n-1}}{c^{(k+1)}_{n-2}}+\frac{{c^{(k+1)}_n}^2}{c^{(k+1)}_{n-1}}+2\frac{c^{(k)}_n}{c^{(k+1)}_n}+\frac{c^{(k+1)}_{n-1}}{{c^{(k+1)}_n}^2}+\frac{c^{(k+2)}_{n-2}}{c^{(k+2)}_{n-1}}+\cdots+\frac{c^{(n)}_k}{c^{(n)}_{k+1}}.\end{aligned} \tag{8.16}$$

The case $k=n$ is easily obtained by the formula in [5, (4.18)]:

$$\Delta_{w_0\Lambda_n,s_k\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=c^{(n)}_n. \tag{8.17}$$

Now, the proof of Theorem 8.1 has been accomplished. □

8.3. **Proof of Theorem 8.2.** The proof of Theorem 8.2 is the same as that of Theorem 7.2 (ii). Indeed, we obtain the formula (7.19) and (7.20) for type B_n. Defining the matrix U_k and Z_k, same as type C_n we have $\det Z_j=1$ $(j=1,\cdots,k-1)$and $\det U_1={c^{(k)}_1}^{-1}$. Thus, we obtain the desired result for type B_n. □

8.4. **Correspondence to the monomial realizations.** Except for $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$ (see 8.5), we shall see the positive answer to the conjecture for type B_n.

First, we see the monomial realization of $B(\Lambda_1)$ associated with the cyclic oder $\cdots(12\cdots n)(12\cdots n)\cdots$, which means that the sign $p_0=(p_{i,j})$ is given by $p_{i,j}=1$ if $i<j$ and $p_{i,j}=0$ if $i>j$ as in the previous section. The crystal $B(\Lambda_1)$ is described as follows: We abuse the notation $B(\Lambda_1):=\{v_i,v_{\overline{i}}|1\leq i\leq n\}\sqcup\{v_0\}$ if there is no confusion. Then the actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1\leq i<n)$ are defined as $\tilde{f}_i=f_i$ and $\tilde{e}_i=e_i$ in (3.6). The actions of $\tilde{f}_n$ and $\tilde{e}_n$ are given as:

$$\tilde{f}_nv_n=v_0,\quad \tilde{f}_nv_0=v_{\overline{n}},\quad \tilde{e}_nv_0=v_n,\quad \tilde{e}_nv_{\overline{n}}=v_0. \tag{8.18}$$

To see the monomial realization $B(c^{(k)}_1)$, we describe the monomials $A_{i,m}$ explicitly:

$$A_{i,m}=\begin{cases}c^{(m)}_1{c^{(m)}_2}^{-1}c^{(m+1)}_1 & \text{for } i=1,\\ c^{(m)}_i{c^{(m)}_{i+1}}^{-1}{c^{(m+1)}_{i-1}}^{-1}c^{(m+1)}_i & \text{for } 1<i<n-1,\\ c^{(m)}_{n-1}{c^{(m)}_n}^{-2}{c^{(m+1)}_{n-2}}^{-1}c^{(m+1)}_{n-1} & \text{for } i=n-1,\\ c^{(m)}_n{c^{(m+1)}_{n-1}}^{-1}c^{(m+1)}_n & \text{for } i=n.\end{cases} \tag{8.19}$$

Here the monomial realization $B(c_1^{(k)})$ for $B(\Lambda_1)$ associated with p_0 is described explicitly:

$$B(c_1^{(k)})=\left\{\frac{{c_j^{(k)}}^{\epsilon_j}}{c_{j-1}^{(k+1)}}=m_1^{(k)}(v_j),\ \frac{c_n^{(k)}}{c_n^{(k+1)}}=m_1^{(k)}(v_0),\ \frac{c_{j-1}^{(k+n-j+1)}}{{c_j^{(k+n-j+1)}}^{\epsilon_j}}=m_1^{(k)}(v_{\bar{j}})\mid 1\le j\le n\right\}, \tag{8.20}$$

where $m_i^{(k)}:B(\Lambda_i)\hookrightarrow\mathcal{Y}(p)$ $(u_{\Lambda_i}\mapsto c_i^{(k)})$ is the embedding of crystal as in Sect.5 and we understand $c_0^{(k)}=1$. Now, Theorem 8.1 and Theorem 8.2 mean the following:

Theorem 8.6. *We obtain* $\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=m_n^{(n)}(u_{\Lambda_n})(=c_n^{(n)})$ *and*

$$\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^n m_1^{(k)}(v_j)+2m_1^{(k)}(v_0)+\sum_{j=k+1}^n m_1^{(k)}(v_{\bar{j}}),\qquad (k=1,2\cdots,n-1) \tag{8.21}$$

$$\Delta_{w_0s_k\Lambda_k,\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^k m_1^{(k-n)}(v_{\bar{j}}),\qquad (k=1,2,\cdots,n-1). \tag{8.22}$$

Note that the second result is derived from the fact that $B(c_1^{(k)})=B({c_1^{(n+k)}}^{-1})$, which is the connected component including ${c_1^{(n+k)}}^{-1}$ as the lowest monomial.

8.5. **Triangles and $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$.** To state the result for $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$, we need certain preparations. The set of *triangles* $\triangle_n$ is defined as follows:

$$\triangle_n:=\{(j_k^{(l)})_{1\le k\le l\le n}|1\le j_k^{(l+1)}\le j_k^{(l)}<j_{k+1}^{(l+1)}\le n+1\quad(1\le k\le l<n)\}. \tag{8.23}$$

We visualize a triangle $(j_k^{(l)})$ in $\triangle_n$ as follows:

$$(j_k^{(l)})=\begin{array}{c} j_1^{(1)}\\ j_2^{(2)}\,j_1^{(2)}\\ j_3^{(3)}\,j_2^{(3)}\,j_1^{(3)}\\ \cdots\cdots\cdots\cdots\\ j_n^{(n)}\,j_{n-1}^{(n)}\cdots j_2^{(n)}\,j_1^{(n)}\end{array}$$

By the definition of $\triangle_n$, we easily obtain

Lemma 8.7. *For any $k\in\{1,2,\cdots,n\}$ there exists a unique j $(1\le j\le k+1)$ such that the kth row of a triangle $(j_k^{(l)})$ in $\triangle_n$ is in the following form:*

$$k\text{-th row}\quad (j_k^{(k)},j_{k-1}^{(k)},\cdots,j_2^{(k)},j_1^{(k)})=(k+1,k,k-1,\cdots,j+1,j-1,j-2,\cdots,2,1), \tag{8.24}$$

that is, we have $j_m^{(k)}=m$ for $m<j$ and $j_m^{(k)}=m+1$ for $m\ge j$.

For a triangle $\delta=(j_k^{(l)})$, we list j's as in the lemma: $s(\delta):=(s_1,s_2,\cdots,s_n)$, which we call the *label* of a triangle δ. Here we have

Lemma 8.8. *For $\delta\in\triangle_n$ let $s(\delta)=(s_1,\cdots,s_n)$ be its label. Then, we have*

(*i*) *The label $s(\delta)$ satisfies $1\le s_k\le k+1$, and $s_{k+1}=s_k$ or s_k+1 for $k=1,\cdots,n$.*

(*ii*) *Each k-th row of a triangle δ is in one of the following* I, II, III, IV:

- I. $s_{k+1}=s_k+1$ *and* $s_k=s_{k-1}$.
- II. $s_{k+1}=s_k$ *and* $s_k=s_{k-1}$.
- III. $s_{k+1}=s_k+1$ *and* $s_k=s_{k-1}+1$.
- IV. $s_{k+1}=s_k$ *and* $s_k=s_{k-1}+1$.

Here we suppose that $s_0=1$ and $s_{n+1}=s_{n-1}+1$, which means that the 1st row must be in I,II or IV *and the n-th row must be in* I or IV.

Now, we associate a Laurant monomial $m(\delta)$ in variables $(c_i^{(j)})_{i\in I,j\in\mathbb{Z}}$ with a triangle $\delta=(j_k^{(l)})$ by the following recipe.

(i) Let $s=(s_1\cdots,s_n)$ be the label of δ.
(ii) If i-th row is in the form I, then associate $c_i^{(s_i)}$.
(iii) If i-th row is in the form IV, then associate $c_i^{(s_i)^{-1}}$.
(iv) If i-th row is in the form II or III, then associate 1.
(v) Take the product of all monomials as above for $1\le i\le n$, then we obtain the monomial $m(\delta)$ associated with δ. This defines a map $m:\triangle_n\to\mathcal{Y}$, where $\mathcal{Y}$ is the set of Laurant monomials in $(c_i^{(j)})_{i\in I,j\in\mathbb{Z}}$.

Let us denote the special triangle $\delta=(j_k^{(l)})$ such that $j_k^{(l)}=k+1$(resp. $j_k^{(l)}=k$) for any k,l by δ_h (reps. δ_l). Then, we can present the result for type B_n.

Theorem 8.9. *For the type B_n, we obtain the explicit form:*

$$\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{\delta\in\triangle_n\setminus\{\delta_l\}}\overline{m(\delta)}, \tag{8.25}$$

where the monomial $\overline{m(\delta)}$ is obtained by applying $\bar{\ }:c_i^{(j)}\to c_i^{(n-j+1)^{-1}}$.

The proof of this theorem will be given in 8.7.

Example 8.10. *The set of triangles $\triangle_4$ is as follows:*

$$\begin{array}{cccccccc}
1&1&1&1&2&2&1&1\\
21&21&21&31&31&31&31&21\\
321&321&421&421&421&421&421&421\\
4321&5321&5321&5321&5321&5421&5421&5421\\
\\
1&2&2&1&2&2&2&2\\
31&31&32&31&31&32&32&32\\
431&431&431&431&431&431&432&432\\
5421&5421&5421&5431&5431&5431&5431&5432
\end{array}$$

and their labels $s(\delta)$ are

$$\begin{array}{cccccccc}
(2,3,4,5), & (2,3,4,4), & (2,3,3,4), & (2,2,3,4), & (1,2,3,4), & (1,2,3,3), & (2,2,3,3), & (2,3,3,3),\\
(2,2,2,3), & (1,2,2,3), & (1,1,2,3), & (2,2,2,2), & (1,2,2,2), & (1,1,2,2), & (1,1,1,2), & (1,1,1,1).
\end{array}$$

Then, we have the corresponding monomials $m(\delta)$:

$$\begin{array}{cccccccc}
\frac{1}{c_4^{(5)}}, & \frac{c_4^{(4)}}{c_3^{(4)}}, & \frac{c_3^{(3)}}{c_2^{(3)}c_4^{(4)}}, & \frac{c_2^{(2)}}{c_1^{(2)}c_4^{(4)}}, & \frac{c_1^{(1)}}{c_4^{(4)}}, & \frac{c_1^{(1)}c_4^{(3)}}{c_3^{(3)}}, & \frac{c_2^{(2)}c_4^{(3)}}{c_1^{(2)}c_3^{(3)}}, & \frac{c_4^{(3)}}{c_2^{(3)}},\\
\frac{c_3^{(2)}}{c_1^{(2)}c_4^{(3)}}, & \frac{c_1^{(1)}c_3^{(2)}}{c_2^{(2)}c_4^{(3)}}, & \frac{c_2^{(1)}}{c_4^{(3)}}, & \frac{c_4^{(2)}}{c_1^{(2)}}, & \frac{c_1^{(1)}c_4^{(2)}}{c_2^{(2)}}, & \frac{c_2^{(1)}c_4^{(2)}}{c_3^{(2)}}, & \frac{c_3^{(1)}}{c_4^{(2)}}, & c_4^{(1)},
\end{array}$$

and we have the monomials $\overline{m(\delta)}$:

$$\begin{array}{cccccccc}
c_4^{(0)}, & \frac{c_3^{(1)}}{c_4^{(1)}}, & \frac{c_2^{(2)}c_4^{(1)}}{c_3^{(2)}}, & \frac{c_1^{(3)}c_4^{(1)}}{c_2^{(3)}}, & \frac{c_4^{(1)}}{c_1^{(4)}}, & \frac{c_3^{(2)}}{c_1^{(4)}c_4^{(2)}}, & \frac{c_1^{(3)}c_3^{(2)}}{c_2^{(3)}c_4^{(2)}}, & \frac{c_2^{(2)}}{c_4^{(2)}},\\
\frac{c_1^{(3)}c_4^{(2)}}{c_3^{(3)}}, & \frac{c_2^{(3)}c_4^{(2)}}{c_1^{(4)}c_3^{(3)}}, & \frac{c_4^{(2)}}{c_2^{(4)}}, & \frac{c_1^{(3)}}{c_4^{(3)}}, & \frac{c_2^{(3)}}{c_1^{(4)}c_4^{(3)}}, & \frac{c_3^{(3)}}{c_2^{(4)}c_4^{(3)}}, & \frac{c_4^{(3)}}{c_3^{(4)}}, & \frac{1}{c_4^{(4)}}.
\end{array}$$

Then, the total sum of all monomials $\overline{m(\delta)}$ except $c_4^{(0)}$ is $\Delta_{w_0s_4\Lambda_4,\Lambda_4}(\Theta^-_{\mathbf{i_0}}(c))$ for B_4.

8.6. **Crystal structure on $\triangle_n$.** Let δ be a triangle. Then the actions of $\tilde{f}_i$ and $\tilde{e}_i$ are defined as follows: Let $J_k = (j_k^{(k)}, j_{k-1}^{(k)}, \cdots, j_2^{(k)}, j_1^{(k)})$ be the k-th row of δ. Thus, we denote $\delta = (J_1, \cdots, J_n)$. It follows from Lemma 8.8 that there exists a unique j such that $J_k = (k+1, k, \cdots, j+1, j-1, \cdots, 2, 1)$ and J_k is in one of I,II,III,IV. Set $J'_k = (k+1, \cdots, j+2, j, j-1, \cdots, 2, 1)$ and $J''_k = (k+1, \cdots, j+1, j, j-2, \cdots, 2, 1)$. Then, we have

$$\tilde{f}_i\delta = \begin{cases} (\cdots, J_{i-1}, J'_i, J_{i+1}, \cdots) & \text{if } J_i \text{ is in I}, \\ \quad 0 & \text{otherwise}, \end{cases} \tag{8.26}$$

$$\tilde{e}_i\delta = \begin{cases} (\cdots, J_{i-1}, J''_i, J_{i+1}, \cdots) & \text{if } J_i \text{ is in IV}, \\ \quad 0 & \text{otherwise}. \end{cases} \tag{8.27}$$

The weight of $\delta = (J_1, \cdots, J_n)$ is defined as follows: Let $s = (s_k)_{k=1,\cdots,n}$ be the label of δ, that is, $J_k = (k+1, k, \cdots, j+1, j-1, \cdots, 2, 1)$ for $j = s_k$:

$$\mathrm{wt}(\delta) = \Lambda_n - \sum_{k=1}^{n}(s_k - 1)\alpha_k. \tag{8.28}$$

We can easily check that $\triangle_n$ is equipped with the crystal structure by (8.26), (8.27) and (8.28), and obtain:

Proposition 8.11. *As a crystal, $\triangle_n$ is isomorphic to $B(\Lambda_n)$. The highest (resp. lowest) weight crystal is δ_h (resp. δ_l) $\in \triangle_n$.*

Proof. As was given in 3.3, we know the explicit form of the crystal $B(\Lambda_n) \cong B_{sp}^{(n)}$. So, let us describe the one-to-one correspondence between $\triangle_n$ and $B_{sp}^{(n)}$: Let $s = (s_1, \cdots, s_n)$ be the label of a triangle $\delta \in \triangle_n$. Now, let us associate $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in B_{sp}^{(n)}$ with s by

$$\epsilon_1 = \begin{cases} + & \text{if } s_1 = 1, \\ - & \text{if } s_1 = 2, \end{cases} \qquad \epsilon_k = \begin{cases} + & \text{if } s_k = s_{k-1}, \\ - & \text{if } s_k = s_{k-1} + 1, \end{cases} (k > 1), \tag{8.29}$$

which defines the map $\psi : \triangle_n \to B_{sp}^{(n)}$. Then, e.g., the vector δ_h (resp. δ_l) corresponds to the highest (resp. lowest) weight vector $(+, +, \cdots, +)$ (resp. $(-, -, \cdots, -)$). It is clear to find that the map ψ is bijective. Now, if k-th row of δ is in type I, then $(s_{k-1}, s_k, s_{k+1}) = (j, j, j+1)$ for some j and in the corresponding $\epsilon = \psi(\delta)$, we have $\epsilon_k = +, \epsilon_{k+1} = -$. For $\tilde{f}_k(\delta)$ let s' be its label and $\epsilon' := \psi(\tilde{f}_k\delta)$. It is easy to see that $(s'_{k-1}, s'_k, s'_{k+1}) = (j, j+1, j+1)$ and $\epsilon'_k = -$, $\epsilon'_{k+1} = +$, which shows that the map ψ is compatible with the action of $\tilde{f}_k$, that is, $\epsilon' = \psi(\tilde{f}_k\delta) = \tilde{f}_k\epsilon = \tilde{f}_k\psi(\delta)$. For the types II,III and IV we can see $\tilde{f}_k(\delta) = 0$ and $\tilde{f}_k\psi(\delta) = 0$, thus we find the compatibility of ψ. By arguing similarly, we can also see the compatibility of ψ with the action of $\tilde{e}_k$'s. Thus, we find that the map ψ is an isomorphism of crystals. □

Next, let us show that the map $m : \triangle_n \to \mathcal{Y}$ gives an isomorphism between $\triangle_n$ and the monomial realization of $B(\Lambda_n)$. Consider the crystal structure on $\mathcal{Y}$ by taking $\overline{p} = (\overline{p}_{i,j})$ such that

$$\overline{p}_{i,j} = \begin{cases} 1 & \text{if } i > j, \\ 0 & \text{if } i < j. \end{cases} \tag{8.30}$$

Indeed, this corresponds to the cyclic sequence of I such as: $\cdots(nn-1\cdots21)(nn-1\cdots21)\cdots$. So, by the prescription in Sect.5 we obtain the monomial realization $\mathcal{Y}(\overline{p})$. Here, by the definition of the map m we can get that $m(\delta_h) = c_n^{(1)}$, which is one of the highest monomials in $\mathcal{Y}(\overline{p})$ with the highest weight Λ_n.

Proposition 8.12. *Let $B(c_n^{(1)})$ be the connected subcrystal of $\mathcal{Y}(\overline{p})$ whose highest monomial is $c_n^{(1)}$ and which is isomorphic to $B(\Lambda_n)$ of type B_n. Then we have $m(\triangle_n) = B(c_n^{(1)})$.*

Proof. In the setting $\mathcal{Y}(\overline{p})$, we have

$$A_{i,m} = \begin{cases} c_1^{(m)} {c_2^{(m+1)}}^{-1} c_1^{(m+1)}, & \text{for } i = 1, \\ c_i^{(m)} {c_{i-1}^{(m)}}^{-1} {c_{i+1}^{(m+1)}}^{-1} c_i^{(m+1)}, & \text{for } 1 < i < n-1, \\ c_{n-1}^{(m)} {c_{n-2}^{(m)}}^{-1} {c_n^{(m+1)}}^{-2} c_{n-1}^{(m+1)}, & \text{for } i = n-1, \\ c_n^{(m)} {c_{n-1}^{(m)}}^{-1} c_n^{(m+1)} & \text{for } i = n. \end{cases} \tag{8.31}$$

For $\delta \in \triangle_n$ and its label $s = (s_1, \cdots, s_n)$, suppose the k-th row of δ is in I ($k \neq 1, n$), that is, there is some j such that $(s_{k-1}, s_k, s_{k+1}) = (j, j, j+1)$, where as we mentioned above $s_0 = 1$ and $s_{n+1} = s_{n-1} + 1$. In this case, the $k-1$(resp. $k+1$)-th row is in II or IV (resp. III or IV). In the case $k \neq, 1, n-1, n$, by the action of $\tilde{f}_k$ we have $(s_{k-1}, s_k, s_{k+1}) = (j, j, j+1) \to (j, j+1, j+1)$. Then we get the k-th row of $\tilde{f}_k(\delta)$ is in IV and the $k-1$ (resp. $k+1$)-th row is in I or III (resp. I or II), i.e., the following four cases:

$$\tilde{f}_k : (s_{k-2}, s_{k-1}, s_k, s_{k+1}, s_{k+2}) = \begin{cases} (j, j, j, j+1, j+1) & \to (j, j, j+1, j+1, j+1), \\ (j-1, j, j, j+1, j+1) & \to (j-1, j, j+1, j+1, j+1), \\ (j, j, j, j+1, j+2) & \to (j, j, j+1, j+1, j+2), \\ (j-1, j, j, j+1, j+2) & \to (j-1, j, j+1, j+1, j+2). \end{cases}$$

In these cases, the types of the $k-1, k, k+1$-th rows and the parts of monomials related to the variables $c_{k-1}^{(l)}, c_k^{(l)}, c_{k+1}^{(l)}$ are turn out to be as follows:

$$\tilde{f}_k : \begin{array}{ccccccc} (\mathrm{II}, \mathrm{I}, \mathrm{IV}) & \to & (\mathrm{I}, \mathrm{IV}, \mathrm{II}) & & \frac{c_k^{(j)}}{c_{k+1}^{(j+1)}} & \to & \frac{c_{k-1}^{(j)}}{c_k^{(j+1)}} \\ (\mathrm{IV}, \mathrm{I}, \mathrm{IV}) & \to & (\mathrm{III}, \mathrm{IV}, \mathrm{II}) & & \frac{c_k^{(j)}}{c_{k-1}^{(j)} c_{k+1}^{(j+1)}} & \to & \frac{1}{c_k^{(j+1)}} \\ (\mathrm{II}, \mathrm{I}, \mathrm{III}) & \to & (\mathrm{I}, \mathrm{IV}, \mathrm{I}) & & c_k^{(j)} & \to & \frac{c_{k-1}^{(j)} c_{k+1}^{(j+1)}}{c_k^{(j+1)}} \\ (\mathrm{IV}, \mathrm{I}, \mathrm{III}) & \to & (\mathrm{III}, \mathrm{IV}, \mathrm{I}) & & \frac{c_k^{(j)}}{c_{k-1}^{(j)}} & \to & \frac{c_{k+1}^{(j+1)}}{c_k^{(j+1)}}. \end{array}$$

In all these cases, the action by $\tilde{f}_k$ on monomial $m(\delta)$ is described as multiplying the monomial $A_{k,j}^{-1} = {c_k^{(j)}}^{-1} c_{k-1}^{(j)} c_{k+1}^{(j+1)} {c_k^{(j+1)}}^{-1}$, which implies $m(\tilde{f}_k \delta) = \tilde{f}_k m(\delta)$.

In the case $k = 1$, if the 1st row is in I, then we get $(s_0, s_1, s_2) = (1, 1, 2)$. This is changed by the action of $\tilde{f}_1$ to $(1, 2, 2)$, which means that as for the labels of the 0th,1st,2nd,3rd rows and the parts of the monomials related to variables $c_1^{(1)}, c_1^{(2)}, c_2^{(2)}$. Hence, we have

$$\tilde{f}_1 : \begin{array}{ccccccc} (\mathrm{I}, \mathrm{IV}) & \to & (\mathrm{IV}, \mathrm{II}) & & \frac{c_1^{(1)}}{c_2^{(2)}} & \to & \frac{1}{c_1^{(2)}}, \\ (\mathrm{I}, \mathrm{III}) & \to & (\mathrm{IV}, \mathrm{I}) & & c_1^{(1)} & \to & \frac{c_2^{(2)}}{c_1^{(2)}}. \end{array}$$

In both cases, the action by $\tilde{f}_1$ is given by multiplying the monomials ${A_{1,1}}^{-1} = {c_1^{(1)}}^{-1} c_2^{(2)} {c_2^{(1)}}^{-1}$.

The cases $k = n$ are also done by the similar way:

$$\tilde{f}_n : (s_{n-2}, s_{n-1}, s_n) = \begin{array}{ccccccc} (j-1, j, j) & \to & (j-1, j, j+1) & & c_{n-1}^{(j)} c_n^{(j)} & \to & \frac{1}{c_n^{(j+1)}} \\ (j, j, j) & \to & (j, j, j+1) & & c_n^{(j)} & \to & \frac{c_{n-1}^{(j)}}{c_n^{(j+1)}}. \end{array}$$

This means that the action by $\tilde{f}_n$ is given by multiplying the monomial ${A_{n,j}}^{-1} = {c_n^{(j)}}^{-1} c_{n-1}^{(j)} {c_n^{(j+1)}}^{-1}$.

Let us see the last case $k = n-1$. If the i-th row of δ is in I, then we have $(s_{n-3}, s_{n-2}, s_{n-1}, s_n) = (j, j, j, j+1)$ or $(j-1, j, j, j+1)$ whose types are (II,I,IV) or (IV,I,IV) respectively if the $(n-1)$-th row is in I. Then we have

$$\tilde{f}_{n-1} : \quad \begin{array}{ccccccc} (\mathrm{II}, \mathrm{I}, \mathrm{IV}) & \to & (\mathrm{I}, \mathrm{IV}, \mathrm{I}) & \qquad & \frac{c_{n-1}^{(j)}}{c_n^{(j+1)}} & \to & \frac{c_{n-2}^{(j)} c_n^{(j+1)}}{c_{n-1}^{(j+1)}}, \\ (\mathrm{IV}, \mathrm{I}, \mathrm{IV}) & \to & (\mathrm{III}, \mathrm{IV}, \mathrm{I}) & & \frac{c_{n-1}^{(j)}}{c_{n-2}^{(j)} c_n^{(j+1)}} & \to & \frac{c_n^{(j+1)}}{c_{n-1}^{(j+1)}}. \end{array}$$

This implies that the action by $\tilde{f}_{n-1}$ is described by multiplying the monomial ${A_{n-1,j}}^{-1} = {c_{n-1}^{(j)}}^{-1} c_{n-2}^{(j)} {c_n^{(j+1)}}^{2} {c_{n-1}^{(j+1)}}^{-1}$. As for the actions of $\tilde{e}_i$, we can show that $m(\tilde{e}_i \delta) = \tilde{e}_i m(\delta)$ by the similar manner to the cases of $\tilde{f}_i$. $\square$

Then, by these results we have

Theorem 8.13. *$B(c_n^{(1)}) = m(\triangle_n)) \cong B(\Lambda_n)$ for type B_n.*

8.7. **Proof of Theorem 8.9.** By the explicit descriptions in (3.9) and (3.10), we know that $e_i^2 = f_i^2 = 0$ on $V_{sp}^{(n)}$. Thus, we can write $\boldsymbol{x}_i(c) := \alpha_i^\vee(c^{-1}) x_i(c) = c^{-h_i}(1 + c \cdot e_i)$ and $\boldsymbol{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) = (1 + c \cdot f_i)c^{-h_i}$ on $V_{sp}^{(n)}$ and then for $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in B_{sp}^{(n)}$

$$\boldsymbol{x}_i(c)\epsilon = \begin{cases} c\epsilon + \epsilon' & \text{if } (\epsilon_i, \epsilon_{i+1}) = (-, +), \\ c^{-1}\epsilon & \text{if } (\epsilon_i, \epsilon_{i+1}) = (+, -), \\ \epsilon & \text{otherwise}, \end{cases} \ (i < n), \qquad \boldsymbol{x}_n(c)\epsilon = \begin{cases} c\epsilon + \epsilon'' & \text{if } \epsilon_n = -, \\ c^{-1}\epsilon & \text{if } \epsilon_n = +, \end{cases} \tag{8.32}$$

where $\epsilon' = (\cdots, \epsilon_{i-1}, +, -, \epsilon_{i+2}, \cdots)$ for $i < n$ and $\epsilon'' = (\cdots, \epsilon_{n-1}, +)$.

For $c^{(i)} = (c_1^{(i)}, \cdots, c_n^{(i)}) \in (\mathbb{C}^\times)^n$ set $X^{(i)}(c^{(i)}) := \boldsymbol{x}_1(c_1^{(i)}) \cdots \boldsymbol{x}_n(c_n^{(i)})$ and $X(c) := X^{(n)}(c^{(n)}) \cdots X^{(1)}(c^{(1)})$ where $c := (c^{(n)}, \cdots, c^{(1)})$.

To calculate $\Delta_{w_0 s_n \Lambda_n, \Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$ explicitly, first let us see $\Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c))$ ($\mathbf{i_0}^{-1} = (n\ n-1 \cdots 21)^n$) since for $g \in U\overline{w}_0 U$ we have

$$\Delta_{w_0 s_i \Lambda_i, \Lambda_i}(g) = \Delta_{w_0\Lambda_i, s_i\Lambda_i}(\eta(g)), \tag{8.33}$$

and $\eta(\Theta^-_{\mathbf{i_0}}(c)) = \Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})$ $(- : c_j^{(i)} \mapsto c_{n-j+1}^{(i)})$. By the above formula and $\omega(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = X(c)$, we have

$$\Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c)) = \langle s_n\epsilon_h, X(c)\epsilon_l \rangle$$

where $s_n\epsilon_h = s_n(+, \cdots, +) = (+, \cdots, +, -)$ and $\epsilon_l = (-, \cdots, -)$. Thus, we would like to see the coefficient of the vector $(+, \cdots, +, -)$ in $X(c)\epsilon_l$.

For the sequence $\mathbf{i_0} = (12 \cdots n)^n$, define J to be the set of all subsequences of $\mathbf{i_0}$. Since each $\boldsymbol{x}_i(c)$ can be written in the form $c^{h_i}(1 + ce_i)$ on V_{sp}, the operator $X(c)$ is expanded in the form:

$$X(c) = \sum_{\xi \in J} \alpha_\xi e_\xi, \tag{8.34}$$

where $e_\xi := e_{i_1} \cdots e_{i_m}$ for $\xi = (i_1, \cdots, i_m) \in J$ and α_ξ is a product of $\alpha_i^\vee(c)$'s and some scalars. Now, we shall find for which subsequence $(i_1, \cdots, i_k) \in J$ we get

$$e_{i_1} \cdots e_{i_k}\epsilon_l = (+, \cdots, +, -). \tag{8.35}$$

Here note that

$$\mathrm{wt}(+, \cdots, +, -) - \mathrm{wt}(\epsilon_l) = \sum_{i=1}^n i\alpha_i - \alpha_n, \tag{8.36}$$

and then for $(i_1, \cdots, i_k)$ in (8.35), we know that if $i \neq n$ the number of i's is i and the number of n is $n-1$.

Let us associate an element in $U(\mathfrak{n}_+)$ with a triangle $\delta = (j_k^{(l)}) \in \triangle_n$ by the following way:

(i) For $i = 1, \cdots, n$ define $r_i(\delta)$ be the set of indices as $r_i(\delta) := \{(k,l) | j_k^{(l)} = i\}$ and $n_i(\delta) := \sharp r_i(\delta)$.

(ii) Set $L = n_i(\delta)$. For $r_i(\delta) = \{(k_{a_1}, l_{a_1}), \cdots, (k_{a_L}, l_{a_L})\}$ define $l_{a_1}, \cdots, l_{a_L}$ with $l_{a_1} < l_{a_2} < \cdots < l_{a_L}$.

(iii) For δ let $r_i(\delta)$ be as in (ii). We set

$$E_i(\delta) := e_{l_{a_1}} e_{l_{a_2}} \cdots e_{l_{a_L}}$$

and $E(\delta) := E_n(\delta) \cdot E_{n-1}(\delta) \cdots E_1(\delta)$.

Example 8.14. *For $n = 4$ and the triangle*

$$\delta = \begin{array}{c} 2 \\ 32 \\ 431 \\ 5421 \end{array} \qquad \textit{we obtain} \qquad \begin{array}{ll} r_1(\delta) = \{(1,3),(1,4)\}, & E_1(\delta) = e_3e_4, \\ r_2(\delta) = \{(1,1),(1,2),(2,4)\}, & E_2(\delta) = e_1e_2e_4 \\ r_3(\delta) = \{(2,2),(2,3)\}, & E_3(\delta) = e_2e_3 \\ r_4(\delta) = \{(3,3),(3,4)\}, & E_4(\delta) = e_3e_4, \end{array}$$

and then $E(\delta) = (e_3e_4)(e_2e_3)(e_1e_2e_4)(e_3e_4)$.

Lemma 8.15. *Let $(j_1, \cdots, j_m) \in J$ be a subsequence of $\mathbf{i_0}$ and $E = e_{j_1} \cdots e_{j_m}$ be the associated monomial of e_i's. Then we have*

(8.37) $\quad E\epsilon_l = (+, \cdots, +, -)$ *if and only if there exists* $\delta \in \triangle_n \setminus \{\delta_l\}$ *such that* $E = E(\delta)$.

Note that $\delta = (j_k^{(l)}) \neq \delta_l$ if and only if $j_n^{(n)} = n+1$.

Proof. First, for $\delta = (j_k^{(l)}) = \delta_l$, that is, $j_k^{(l)} = k$, we have

$$E(\delta_l) = e_n(e_{n-1}e_n)(e_{n-2}e_{n-1}e_n) \cdots (e_1e_2 \cdots e_{n-1}e_n),$$

and then $E(\delta_l)\epsilon_l = (+,+,\cdots,+,+) \neq (+,+,\cdots,+,-)$.

Next, assuming that there exists $\delta \in \triangle_n \setminus \{\delta_l\}$ such that $E = E(\delta)$, show that $E\epsilon_l = s_n\epsilon_h = (+,\cdots,+,-)$. Indeed, if $E\delta \neq 0$, we have that $E\epsilon_l = s_n\epsilon_h$ by the explicit actions of e_i's and the weight counting. So it suffices to show $E\epsilon_l \neq 0$. Now, for $\delta = (j_i^{(k)})$ write

$$E = E(\delta) = e_{l_m} \cdots e_{l_2} e_{l_1} \quad (m = \frac{n(n+1)}{2} - 1).$$

Let us show $e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$ $(1 \leq k \leq m)$ by the induction on k. By the definition of $E(\delta)$ we know that $e_{l_1} = e_n$ and then this implies $e_{l_1}\epsilon_l = e_n\epsilon_l \neq 0$. Assume $e_{l_{k-1}} \cdots e_{l_1}\epsilon_l \neq 0$ and e_{l_k} appears in $E_j(\delta)$. Set $i = l_k$. If $i \leq n-1$, then the $i-1, i, i+1$th row of the triangle δ around this j is in one of the following forms:

$$(a)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!1 & \\ i\text{-th row} & & ⓙ & j\!-\!1 \\ i\!+\!1\text{-th row} & & j & j\!-\!1 \end{array} \quad (b)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!1 & \\ i\text{-th row} & & ⓙ & j\!-\!1 \\ i\!+\!1\text{-th row} & & j & j\!-\!2 \end{array} \quad (c)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!1 & \\ i\text{-th row} & & ⓙ & j\!-\!2 \\ i\!+\!1\text{-th row} & & j & j\!-\!2 \end{array}$$

$$(d)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!2 & \\ i\text{-th row} & & ⓙ & j\!-\!2 \\ i\!+\!1\text{-th row} & & j & j\!-\!2 \end{array} \quad (e)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!1 & \\ i\text{-th row} & & ⓙ & j\!-\!2 \\ i\!+\!1\text{-th row} & & j\!-\!1 & ** \end{array} \quad (f)\ \begin{array}{llll} i\!-\!1\text{-th row} & * & j\!-\!2 & \\ i\text{-th row} & & ⓙ & j\!-\!2 \\ i\!+\!1\text{-th row} & & j\!-\!1 & ** \end{array},$$

where $*$ means j or $j+1$, $**$ means $j-2$ or $j-3$ and ⓙ is the entry which gives $e_{l_k} = e_i$ in $E(\delta)$. As for (a), we see that $E_j = \cdots e_ie_{i+1} \cdots$ and $E_{j-1} = \cdots e_{i-1}e_ie_{i+1} \cdots$, and then we have $e_{l_{k-1}} = e_{i+1}$. Suppose that $j-1$ in the $i+1$-th row gives $e_{i+1} = e_{l_p}$ $(1 \leq p \leq m)$ in $E_{j-1}(\delta)$. Since $e_{l_{k-1}} \cdots e_{l_1}\epsilon_l \neq 0$, we know that the vector before applying e_{i+1} in E_{j-1}, say $\epsilon' = (\epsilon'_i)$, is not zero, that is,

$$\epsilon' = e_{l_{p-1}} \cdots e_{l_1}\epsilon_l \neq 0.$$

Since the vector $e_{i-1}e_ie_{i+1}\epsilon' \neq 0$, the vector ϵ' satisfies $\epsilon'_{i-1} = \epsilon'_i = \epsilon'_{i+1} = -, \epsilon'_{i+2} = +$. If $i \leq n-2$, there exists j in $i+2$-th row of δ and this means that there is e_{i+2} in $E_j(\delta)$ since there are $j-1$ and j in the $i+1$th row. Then, we know that the vector before being applied $e_{l_{k-1}} = e_{i+1}$, say $\epsilon'' = (\epsilon''_i)$, has the signs $\epsilon''_i = \epsilon''_{i+1} = -$ and $\epsilon''_{i+2} = +$, which implies $e_ie_{i+1}\epsilon'' \neq 0$:

$$\begin{aligned}&(\epsilon'_{i-1}, \epsilon'_i, \epsilon'_{i+1}, \epsilon'_{i+2}) = (-,-,-,+) \xrightarrow{e_{i+1}} (-,-,+,-) \xrightarrow{e_i} (-,+,-,-) \xrightarrow{e_{i-1}} (+,-,-,-)\\ &\cdots \xrightarrow{e_{i+2}} (*,-,-,+) \xrightarrow{e_{i+1}} (*,-,+,-) \xrightarrow{e_i} (*,+,-,-).\end{aligned}$$

Thus, we know that $e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$ in the case (a). For the case (b), since we can deduce that there is only $j-1$ or j in the $i+2$th row, the following two cases can occur:

$$E_j(\delta)E_{j-1}(\delta) = (\cdots e_ie_{i+1}\cdots)(\cdots e_{i-1}e_ie_{i+2}\cdots), \text{or } E_j(\delta)E_{j-1}(\delta) = (\cdots e_ie_{i+1}e_{i+2}\cdots)(\cdots e_{i-1}e_i\cdots),$$

where in the R.H.S. of the above equations "$\cdots$" means that there is no e_{i-1}, e_i, e_{i+1} or e_{i+2}. In both cases, before applying e_i in $E_j(\delta)$, we applied e_{i-1} in $E_{j-1}(\delta)$ and then e_{i+1} in $E_j(\delta)$, and we know that the resulting vector ϵ does not vanish by the induction hypothesis and is in the form $\epsilon = (\epsilon_k)$ such that $\epsilon_i = -$ and $\epsilon_{i+1} = +$, which means $e_i\epsilon \neq 0$.

For the cases (c)–(f), arguing similarly we obtain

(c) $E_j(\delta) \cdot E_{j-1}(\delta) = (\cdots e_ie_{i+1}\cdots)(\cdots e_{i-1}e_{i+2}\cdots)$ or $(\cdots e_ie_{i+1}e_{i+2}\cdots)(\cdots e_{i-1}\cdots)$.
(d) $E_j(\delta) \cdot E_{j-1}(\delta) = (\cdots e_ie_{i+1}\cdots)(\cdots e_{i+2}\cdots)$ or $(\cdots e_ie_{i+1}e_{i+2}\cdots)(\cdots)$.
(e) $E_j(\delta) \cdot E_{j-1}(\delta) = (\cdots e_i\cdots)(\cdots e_{i-1}e_{i+1}e_{i+2}\cdots)$ or $(\cdots e_i\cdots)(\cdots e_{i-1}e_{i+1}\cdots)$.
(f) $E_j(\delta) \cdot E_{j-1}(\delta) = (\cdots e_i\cdots)(\cdots e_{i+1}e_{i+2}\cdots)$ or $(\cdots e_i\cdots)(\cdots e_{i+1}\cdots)$.

Thus, in all cases we find that $e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$. The case $i = n-1$ is done by the similar way. So, finally we assume that $i = l_k = n$. Then, the $n-1$-th row and the n-th row of δ around j are as follows:

$$(i)\ \begin{array}{ccc} n{-}1\text{-th row} & * & j{-}1 \\ n\text{-th row} & j & j{-}1 \end{array} \qquad (ii)\ \begin{array}{ccc} n{-}1\text{-th row} & * & j{-}1 \\ n\text{-th row} & j & j{-}2 \end{array} \qquad (iii)\ \begin{array}{ccc} n{-}1\text{-th row} & * & j{-}2 \\ n\text{-th row} & j & j{-}2 \end{array}$$

where $*$ means j or $j+1$ and j is the entry which gives $e_{l_k} = e_n$. As for (i), we know that $E_j(\delta) = \cdots e_n \cdots$ and $E_{j-1}(\delta) = \cdots e_{n-1}e_n\cdots$. Suppose that $j-1$ in the n-th row gives $e_n = e_{l_p}$ in $E_{j-1}(\delta)$. Denote the vector before being applied $e_n = e_{l_p}$ by $\epsilon' = (\epsilon'_i)$, that is, $\epsilon' = e_{l_{p-1}} \cdots e_{l_1}\epsilon_l$, which satisfies $\epsilon'_{n-1} = \epsilon'_n = -$ and then

$$E_{j-1}(\delta) : (\epsilon'_{n-1}, \epsilon'_n) = (-,-) \xrightarrow{e_n} (-,+) \xrightarrow{e_{n-1}} (+,-).$$

This means that the vector before being applied $e_n = e_{l_k}$, say $\epsilon'' = (\epsilon''_i)$, has the form $\epsilon''_n = -$ and then we have $e_n\epsilon'' = e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$. Considering similarly, we have

(ii) $E_j(\delta) \cdot E_{j-1}(\delta) \cdot E_{j-2}(\delta) = (\cdots e_n \cdots)(\cdots e_{n-1}\cdots)(\cdots e_n\cdots)$.
(iii) $E_j(\delta) \cdot E_{j-1}(\delta) \cdot E_{j-2}(\delta) = (\cdots e_n \cdots)(\cdots\cdots)(\cdots e_{n-1}e_n\cdots)$.

These imply that in the both cases the vector applied e_n in $E_j(\delta)$ never vanish. Thus, we also find that $e_n\epsilon'' = e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$ and then $E(\delta)\epsilon_l \neq 0$. Now, we find that $E(\delta)\epsilon_l \neq 0$ for any $\delta \in \Delta_n \setminus \{\delta_h\}$.

Next, assume that $E\epsilon_l = (-,-,\cdots,-,+)$. Under this assumption, by (8.36) we have that each e_i $(i \neq n)$ should appear i-times in E and e_n should appear $n-1$-times in E. In the sequence $\mathbf{i_0}$ there are n cycles $(12\cdots n)$ just as

$$\mathbf{i_0} = \underbrace{12\cdots n}_{n\text{th cycle}} \quad \underbrace{12\cdots n}_{n-1\text{th cycle}} \quad \cdots \quad \underbrace{12\cdots n}_{2\text{nd cycle}} \quad \underbrace{12\cdots n}_{1\text{st cycle}}$$

Since there are n positions of the index n in $\mathbf{i_0}$, we should choose $n-1$ from them, which is written as $(n, n-1, \cdots, j+1, j-1, \cdots, 2, 1)$ for some j. This defines $(j^{(n)}_{n-1}, j^{(n)}_{n-2}, \cdots, j^{(n)}_2, j^{(n)}_1)$, which is the bottom row(= n-th row) of an element in $\triangle_n$ except the first entry $j^{(n)}_n$. Here, we fix $j^{(n)}_n = n+1$. Next, let us see e_{n-1}. We can apply e_{n-1} after applying e_n and can apply e_n after applying e_{n-1}.

Thus, we get the positions of $n-1$'s in $\mathbf{i_0}$, which must be in between two n's chosen in the previous step. Then, we obtain the list $(j^{(n-1)}_{n-1}, j^{(n-1)}_{n-2}, \cdots, j^{(n-1)}_2, j^{(n-1)}_1)$, which should satisfy the condition $j^{(n)}_k \le j^{(n-1)}_k < j^{(n)}_{k+1}$ $(1 \le k \le n-1)$ and becomes the $n-1$th row of the corresponding δ. This condition just coincides with the ones for $\triangle_n$. Repeating these steps, we obtain a triangle δ and find that there exists $\delta \in \triangle_n$ such that $E = E(\delta)$. □

To show Theorem 8.9, let us find the coefficient of the vector generated from ϵ_l by applying $a_\xi e_{i_1} \cdots e_{i_k}$ in (8.34). Set $\mathbf{E}_i(c) = c\alpha_i^\vee(c^{-1})e_i$. Then we can write $\boldsymbol{x}_i(c) = \alpha_i^\vee(c^{-1}) + \mathbf{E}_i(c)$. Here note that for $\epsilon = (\epsilon_i) \in B^{(n)}_{sp}$:

$$\mathbf{E}_i(c)\epsilon = c\alpha_i^\vee(c^{-1})e_i(\epsilon_1, \cdots, \epsilon_n) = \begin{cases} (\cdots, \overset{i}{+}, \overset{i+1}{-}, \cdots) & \text{if } \epsilon_i = -,\ \epsilon_{i+1} = +,\ i \neq n, \\ (\cdots\cdots\cdots, \overset{n}{+}) & \text{if } \epsilon_n = -,\ i = n, \\ \quad 0 & \text{otherwise.} \end{cases}$$

We consider the four cases I-IV: If the i-th row of δ is in I, there exists j such that the $i-1$, i, $i+1$th rows of δ are in the form:

$$\begin{array}{llllllll} i-1\text{-th row} & \cdots j+3 & j+2 & j+1 & j-1 & j-2 & & \\ i\text{-th row} & \cdots & j+3 & j+2 & j+1 & j-1 & j-2 & \\ i+1\text{-th row} & \cdots & & j+3 & j+2 & j & j-1 & j-2 \end{array}$$

This means that in the expansion (8.34) we know that this δ gives the following monomial in which $\alpha_i^\vee$ appears once:

$$\cdots \alpha_i^\vee({c_i^{(j)}}^{-1})\mathbf{E}_{i+1}(c^{(j)}_{i+1}) \cdots \mathbf{E}_{i-1}(c^{(j-1)}_{i-1})\mathbf{E}_i(c^{(j-1)}_i)\mathbf{E}_{i+1}(c^{(j-1)}_{i+1}) \cdots \tag{8.38}$$

Thus, by the action of $\alpha_i^\vee({c_i^{(j)}}^{-1})$ we obtain the coefficient $c_i^{(j)}$. In the other cases II,III,IV, discussing similarly we obtain the coefficient 1 for II and III, and we have the coefficient ${c_i^{(j)}}^{-1}$ for IV, which coincides with the recipe after Lemma 8.8 and then we know that the desired coefficient is the same as $m(\delta)$.

As has been shown above, the set of monomials $m(\triangle_n)$ has the crystal structure isomorphic to $B(c_n^{(1)}) (\cong B(\Lambda_n))$. The following lemma is direct from Theorem 5.4.

Lemma 8.16. *The set of monomials $\overline{m(\triangle_n)}$ is a crystal isomorphic to $B(c_n^{(0)}) \cong B(\Lambda_n)$.*

Proof. This is the case $a = n+1$ in Theorem 5.4. Indeed, the explicit form of $A_{i,l}$ is given in (8.31), which induces the formula $\overline{A^{-1}_{i,l}} = A_{i,n-l}$. □

Thus, by the formula (8.33), we obtain

$$\Delta_{w_0 s_n \Lambda_n, \Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})),$$

where for $c = (c^{(1)}, \cdots, c^{(n)})$ $(c^{(i)} = (c^{(i)}_l)_{1 \le l \le n})$ we define $\overline{c} = (\overline{c^{(n)}}, \cdots, \overline{c^{(1)}})$ $(\overline{c^{(i)}} = ({c^{(i)}_{n-l+1}}^{-1})_{1 \le l \le n})$. Therefore, we have completed the proof of Theorem 8.9. □

Hence, we obtain the affirmative answer to our conjecture for type B_n.

9. Explicit form of $f_B(t\Theta^-_{\mathbf{i_0}}(c))$ for D_n

9.1. **Main Theorems.** In case of type D_n, fix the cyclic reduced longest word $\mathbf{i_0} = (12 \cdots n{-}1\, n)^{n-1}$.

Theorem 9.1. *For $k \in \{1, 2, \cdots, n\}$ and $c = (c_i^{(j)}) = (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(n-1)}, c_n^{(n-1)}) \in (\mathbb{C}^\times)^{n(n-1)}$, we have*

$$\begin{aligned}
&\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) \\
&= c_1^{(k)} + \frac{c_2^{(k)}}{c_1^{(k+1)}} + \cdots + \frac{c_{n-2}^{(k)}}{c_{n-3}^{(k+1)}} + \frac{c_{n-1}^{(k)} c_n^{(k)}}{c_{n-2}^{(k+1)}} + \frac{c_{n-2}^{(k+1)}}{c_{n-1}^{(k+1)} c_n^{(k+1)}} + \frac{c_n^{(k)}}{c_{n-1}^{(k+1)}} + \frac{c_{n-1}^{(k)}}{c_n^{(k+1)}} + \frac{c_{n-3}^{(k+2)}}{c_{n-2}^{(k+2)}} + \cdots + \frac{c_k^{(n-1)}}{c_{k+1}^{(n-1)}} \\
&\qquad (k = 1, 2, \cdots, n-2), \\
&\Delta_{w_0\Lambda_{n-1}, s_{n-1}\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c)) = c_{n-1}^{(n-1)}, \qquad \Delta_{w_0\Lambda_n, s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = c_n^{(n-1)}.
\end{aligned}$$

Theorem 9.2. *Let k be in $\{1, 2, \cdots, n-2\}$. Then we have*

$$\Delta_{w_0 s_k\Lambda_k, \Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \frac{1}{c_1^{(k)}} + \sum_{j=1}^{k-1} \frac{c_{k-j}^{(j)}}{c_{k-j+1}^{(j)}}. \tag{9.1}$$

The proof of Theorem 9.2 is the almost same as the ones for Theorem 7.2 and Theorem 8.2.

The cases $k = n-1, n$ will be presented in 9.4. We shall prove the above theorems in the next section.

9.2. **Proof of Theorem 9.1.** Considering similarly to type C_n as in 7.2, we can write

$$\begin{aligned}
\boldsymbol{x}_i(c) := \alpha_i^\vee(c^{-1})x_i(c) &= c^{-h_i}(1 + c \cdot e_i), \\
\boldsymbol{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) &= (1 + c \cdot f_i)c^{-h_i},
\end{aligned}$$

since $f_i^2 = e_i^2 = 0$ on the vector representation $V(\Lambda_1)$. We also have $\omega(\boldsymbol{y}_i(c)) = \alpha_i^\vee(c^{-1})x_i(c) = \boldsymbol{x}_i(c)$ and define the coefficients ${}^{\bar{i}}\Xi_j^{(p)} = {}^{\bar{i}}\Xi_j^{(p)}(c^{[1:p]})$ and ${}^{i}\Xi_j^{(p)} = {}^{i}\Xi_j^{(p)}(c^{[1:p]})$ for $j \in I$ and $p \in \{1, 2, \cdots, n-1\}$ by

$$\begin{aligned}
X^{(p)}X^{(p-1)} \cdots X^{(1)}v_i &= \sum_{j=1}^n {}^{i}\Xi_j^{(p)} v_j + \sum_{j=1}^n {}^{i}\Xi_{\bar{j}}^{(p)} v_{\bar{j}} \in V(\Lambda_1) \quad (i = 1, \cdots, n), \\
X^{(p)}X^{(p-1)} \cdots X^{(1)}v_{\bar{i}} &= \sum_{j=1}^n {}^{\bar{i}}\Xi_j^{(p)} v_j + \sum_{j=1}^n {}^{\bar{i}}\Xi_{\bar{j}}^{(p)} v_{\bar{j}} \in V(\Lambda_1) \quad (i = 1, 2, \cdots, n),
\end{aligned}$$

where $c^{[1:p]} = (c_1^{(1)}, c_2^{(1)}, \cdots, c_{n-1}^{(p)}, c_n^{(p)})$ and $X^{(p)} = \boldsymbol{x}_n(c_n^{(p)})\boldsymbol{x}_{n-1}(c_{n-1}^{(p)}) \cdots \boldsymbol{x}_1(c_1^{(p)})$.

It follows from (6.3) and $\omega(\Theta_{\mathbf{i}_0}(c)) = X^{(n-1)} \cdots X^{(1)}$ that we also have

$$\Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i}}(c)) = \langle \overline{s}_i \cdot u_{\Lambda_i}\,,\, X^{(n-1)} \cdots X^{(1)}v_{\Lambda_i}\rangle, \tag{9.2}$$

which is almost same as (7.4). To describe ${}^{\bar{i}}\Xi_j^{(p)}$ explicitly let us define the *segments* of type D_n, which are similar to the ones for B_n and C_n. For $k \in I$ and $p \in \{1, 2, \cdots, n-1\}$ set $L := p - n + k + 1$ and $S := n - k$, which are slightly different from the other types. For $M \in \mathcal{M}_k^{(p)}[r] := \{M = \{m_2, m_3, \cdots, m_L\} | 2 + r \leq m_2 < \cdots m_L \leq p + r\}$ $(r \in \mathbb{Z})$. As in the previous sections, we denote $\mathcal{M}_k^{(p)}[0]$ by $\mathcal{M}_k^{(p)}$.

For $m \in M_j \subset M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}$, define $n(m) := n - j + 1$. For $M = M_1 \sqcup \cdots \sqcup M_S \in \mathcal{M}_k^{(p)}$, write $M_1 = \{2, 3, \cdots, a\}$. For $1 \leq b \leq c \leq a \leq p$ and $i \leq i_2 \leq \cdots \leq i_b \leq n-1$ define the monomials

in $(c_j^{(i)})_{1\le i<n,1\le j\le n}$,

$$C^M_{i_2,i_3,\cdots,i_b}:=\frac{(c^{(1)}_{i_2}\cdots c^{(b-1)}_{i_b})({c_n^{(1)}}^{\epsilon_{i_2+1}}\cdots {c_n^{(b-1)}}^{\epsilon_{i_b+1}})}{c^{(2)}_{i_2-1}\cdots c^{(b)}_{i_b-1}},\qquad D^M:=\prod_{m\in M\setminus M_1}\frac{c^{(m)}_{n(m)-1}}{c^{(m)}_{n(m)}}, \tag{9.3}$$

$$E^M_{b,c}:=\frac{c_n^{(b)}c_n^{(b+1)}\cdots c_n^{(c-1)}}{c^{(b+1)}_{n-1}c^{(b+2)}_{n-1}\cdots c^{(c)}_{n-1}}+\frac{c^{(b)}_{n-1}c^{(b+1)}_{n-1}\cdots c^{(c-1)}_{n-1}}{c_n^{(b+1)}c_n^{(b+2)}\cdots c_n^{(c)}}, \tag{9.4}$$

$$F^M_{c,a}:=\frac{c^{(c+1)}_{n-2}c^{(c+2)}_{n-2}\cdots c^{(a)}_{n-2}}{(c^{(c+1)}_{n-1}c^{(c+1)}_{n})(c^{(c+2)}_{n-1}c^{(c+2)}_{n})\cdots(c^{(a)}_{n-1}c^{(a)}_{n})}, \tag{9.5}$$

where $\epsilon_i=\delta_{i,n}$ and $C^M_{i_2,i_3,\cdots,i_a}=1$ (resp. $D^M=1$, $E^M_{b,c}=1$, $F^M_{c,a}=1$) if $M_1=\emptyset$ or $b=1$ (resp. $M\setminus M_1=\emptyset$, $M_1=\emptyset$ or $b=c$, $M_1=\emptyset$ or $c=a$).

Proposition 9.3. *In the setting above, we have*

$${}^{\bar{i}}\Xi^{(p)}_{\bar{k}}=\sum_{i\le i_2\le\cdots\le i_p\le k}\frac{(c^{(1)}_{i_2}c^{(2)}_{i_3}\cdots c^{(p-1)}_{i_p}c^{(p)}_k{c_n^{(p)}}^{\epsilon'_k}){c_n^{(1)}}^{\epsilon'_{i_2}}{c_n^{(2)}}^{\epsilon'_{i_3}}\cdots{c_n^{(p-1)}}^{\epsilon'_{i_p}}}{c^{(1)}_{i-1}c^{(2)}_{i_2-1}\cdots c^{(p)}_{i_p-1}} \tag{9.6}$$
$$(k=1,2,\cdots,n),$$

$${}^{\bar{i}}\Xi^{(p)}_{n}=\sum_{\substack{i\le i_2\le\cdots\le i_p\le n+1\\ i_2,\cdots,i_p\ne n}}\frac{(c^{(1)}_{i_2-2\epsilon''_{i_2}}c^{(2)}_{i_3-2\epsilon''_{i_3}}\cdots c^{(p-1)}_{i_p-2\epsilon''_{i_p}}c^{(p)}_{n-1}){c_n^{(1)}}^{\epsilon'_{i_2}}{c_n^{(2)}}^{\epsilon'_{i_3}}\cdots{c_n^{(p-1)}}^{\epsilon'_{i_p}}}{c^{(1)}_{i-1}c^{(2)}_{i_2-1}\cdots c^{(p)}_{i_p-1}} \tag{9.7}$$

$${}^{\bar{i}}\Xi^{(p)}_{k}={c^{(1)}_{i-1}}^{-1}\sum_{(A)}C^M_{i_2,\cdots,i_b}\cdot D^M\cdot E^M_{b,c}\cdot F^M_{c,a}\qquad(k=1,2,\cdots,n-1), \tag{9.8}$$

where $\epsilon'_i=\delta_{i,n-1}$, $\epsilon''_i=\delta_{i,n+1}$ and the condition (A) is as follows:

(A) $M=M_1\sqcup\cdots\sqcup M_S\in\mathcal{M}^{(p)}_k$, $i\le i_2\le\cdots\le i_b\le n-1$, $M_1=\{2,\cdots,a\}$, $1\le b<c\le a$.

Proof. Set $\mathcal{X}:=\boldsymbol{x}_n(c_n)\cdots\boldsymbol{x}_1(c_1)$. By calculating directly we have the formula:

$$\mathcal{X}v_i=\begin{cases}c_1^{-1}v_1 & \text{if } i=1,\\ c_{i-1}c_i^{-1}v_i+v_{i-1} & \text{if } i=2,\cdots,n-2,\\ c_{n-2}c_{n-1}^{-1}c_n^{-1}v_{n-1}+v_{n-2} & \text{if } i=n-1,\\ c_{n-1}c_n^{-1}v_n+c_n^{-1}v_{n-1} & \text{if } i=n,\end{cases} \tag{9.9}$$

$$\mathcal{X}v_{\bar{i}}=\begin{cases}c_{i-1}^{-1}(c_iv_{\bar{i}}+\cdots+c_{n-2}v_{\overline{n-2}}+c_{n-1}c_nv_{\overline{n-1}}+c_nv_{\bar{n}}+c_{n-1}v_n+v_{n-1}) & \text{if } i<n,\\ c_nc_{n-1}^{-1}v_{\bar{n}}+c_{n-1}^{-1}v_{n-1} & \text{if } i=n.\end{cases} \tag{9.10}$$

where we understand $c_0 = 1$. Using these, we get

$$ {}^{\bar{i}}\Xi^{(p)}_{\bar{k}} = \sum_{j=i}^{k} {}^{\bar{i}}\Xi^{(p-1)}_{\bar{j}} \frac{c^{(p)}_k {c^{(p)}_n}^{\epsilon'_k}}{c^{(p)}_{j-1}}, \quad (k = 1, \cdots, n), \tag{9.11} $$

$$ {}^{\bar{i}}\Xi^{(p)}_{k} = {}^{\bar{i}}\Xi^{(p-1)}_{k+1} + {}^{\bar{i}}\Xi^{(p-1)}_{k} \frac{c^{(p)}_{k-1}}{c^{(p)}_k}, \quad (k = 1 \cdots, n-2), \tag{9.12} $$

$$ {}^{\bar{i}}\Xi^{(p)}_{n-1} = \sum_{j=i}^{n} {}^{\bar{i}}\Xi^{(p-1)}_{\bar{j}} {c^{(p)}_{j-1}}^{-1} + {}^{\bar{i}}\Xi^{(p-1)}_{n} {c^{(p)}_n}^{-1} + {}^{\bar{i}}\Xi^{(p-1)}_{n-1} \frac{c^{(p)}_{n-2}}{c^{(p)}_{n-1} c^{(p)}_n}, \tag{9.13} $$

$$ {}^{\bar{i}}\Xi^{(p)}_{n} = \left(\sum_{j=i}^{n-1} {}^{\bar{i}}\Xi^{(p-1)}_{\bar{j}} {c^{(p)}_{j-1}}^{-1} c^{(p)}_{n-1} \right) + {}^{\bar{i}}\Xi^{(p-1)}_{n} c^{(p)}_{n-1} {c^{(p)}_n}^{-1}. \tag{9.14} $$

Indeed, the formulae (9.6) and (9.7) are easily proved by the induction on p using the formulae (9.11) and (9.14) as the other types.

Considering the segments in $\mathcal{M}^{(p)}_k$, $\mathcal{M}^{(p-1)}_{k+1}$ and $\mathcal{M}^{(p-1)}_k$ as the cases for C_n and applying the recursions (9.12) and (9.13) to the induction hypothesis, we obtain (9.8). □

Thus, for example, we have

$$ \Delta_{w_0\Lambda_1, s_1\Lambda_1}(\Theta^-_{\mathbf{i_0}}(c)) = {}^{\bar{1}}\Xi^{(n-1)}_{2} = \sum_{j=1}^{n-2} \frac{c^{(1)}_j}{c^{(2)}_{j-1}} + \frac{c^{(1)}_{n-1} c^{(1)}_n}{c^{(2)}_{n-2}} + \frac{c^{(1)}_n}{c^{(2)}_{n-1}} + \frac{c^{(1)}_{n-1}}{c^{(2)}_n} + \frac{c^{(2)}_{n-2}}{c^{(2)}_{n-1} c^{(2)}_n} + \sum_{j=3}^{n-1} \frac{c^{(j)}_{n-j}}{c^{(j)}_{n-j+1}}. $$

The following is similar to Lemma 7.6 and Lemma 8.4.

Lemma 9.4. *For $k = 1, \cdots, n-2$ we define the matrix W_k by*

$$ W_k := \begin{pmatrix} {}^{\bar{1}}\Xi^{(n-1)}_{k+1} & {}^{\bar{1}}\Xi^{(n-1)}_{k-1} & \cdots & {}^{\bar{1}}\Xi^{(n-1)}_{2} & {}^{\bar{1}}\Xi^{(n-1)}_{1} \\ {}^{\bar{2}}\Xi^{(n-1)}_{k+1} & {}^{\bar{2}}\Xi^{(n-1)}_{k-1} & \cdots & {}^{\bar{2}}\Xi^{(n-1)}_{2} & {}^{\bar{2}}\Xi^{(n-1)}_{1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ {}^{\bar{k}}\Xi^{(n-1)}_{k+1} & {}^{\bar{k}}\Xi^{(n-1)}_{k-1} & \cdots & {}^{\bar{k}}\Xi^{(n-1)}_{2} & {}^{\bar{k}}\Xi^{(n-1)}_{1} \end{pmatrix}. \tag{9.15} $$

Then we have $\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c)) = \det W_k$.

Similar to the previous cases, the last column of the matrix W_k is given as

$$ {}^t({}^{\bar{1}}\Xi^{(n-1)}_1, \cdots, {}^{\overline{k-1}}\Xi^{(n-1)}_1) = {}^t(1, {c^{(1)}_1}^{-1}, {c^{(1)}_2}^{-1}, \cdots, {c^{(1)}_{k-1}}^{-1}). $$

Then, applying the elementary transformations on W_k by (i-th row) $-\frac{c^{(1)}_i}{c^{(1)}_{i-1}} \times$($i+1$-th row) for $i = 1, \cdots, k-1$, in the transformed matrix $\tilde{W}_k$ its (i,j)-entry is as follows:

Lemma 9.5. *For $c = (c^{(l)}_k | 1 \le k \le n, 1 \le l < n)$ we set $c^{(l)} := (c^{(l)}_1, \cdots, c^{(l)}_n)$ and $c^{[a:b]} := (c^{(a)}, c^{(a+1)}, \cdots, c^{(b)})$ $(a \le b)$. For $i = 1, \cdots, k-1$ the (i,j)-entry $(\tilde{W}_k)_{i,j}$ is:*

$$ (\tilde{W}_k)_{i,j} = \begin{cases} {}^{\bar{i}}\Xi^{(n-2)}_{k+1}(c^{[2:n-1]}) & \text{if } j = 1, \\ {}^{\bar{i}}\Xi^{(n-2)}_{k-j+1}(c^{[2:n-1]}) & \text{if } 1 < j < k, \\ 0 & \text{if } j = k, \end{cases} \tag{9.16} $$

Note that for type D_n by definition we have $c = c^{[1:n-1]}$.

Proof. The proof is similar to the one for Lemma 7.7. Indeed, for type D_n we also get the same formula as (7.15):

$$ {}^{\bar{i}}\Xi_j^{(n-1)}(c) - \frac{c_i^{(1)}}{c_{i-1}^{(1)}} {}^{\overline{i+1}}\Xi_j^{(n-1)}(c) = \frac{c_i^{(1)}}{c_{i-1}^{(1)}} {}^{\bar{i}}\Xi_j^{(n-2)}(c^{[2,n-1]}). $$

Then, this proves the lemma. $\square$

Applying the above elementary transformations to the matrix W_k, we find

$$ \det W_k = {}^{\bar{1}}\Xi_{k+1}^{(n-k)}(c^{[k:n-1]}) $$

Thus, it follows from (9.8) that for $k = 1, 2, \cdots, n-2$

$$ \begin{aligned} &\Delta_{w_0\Lambda_k, s_k\Lambda_k}(\Theta_{\mathbf{i}}^-(c)) = {}^{\bar{1}}\Xi_{k+1}^{(n-k)}(c^{[k:n-1]}) \\ &= c_1^{(k)} + \frac{c_2^{(k)}}{c_1^{(k+1)}} + \cdots + \frac{c_{n-2}^{(k)}}{c_{n-3}^{(k+1)}} + \frac{c_{n-1}^{(k)}c_n^{(k)}}{c_{n-2}^{(k+1)}} + \frac{c_{n-2}^{(k+1)}}{c_{n-1}^{(k+1)}c_n^{(k+1)}} + \frac{c_n^{(k)}}{c_{n-1}^{(k+1)}} + \frac{c_{n-1}^{(k)}}{c_n^{(k+1)}} + \frac{c_{n-3}^{(k+2)}}{c_{n-2}^{(k+2)}} + \cdots + \frac{c_k^{(n-1)}}{c_{k+1}^{(n-1)}}. \end{aligned} $$

The cases $k = n-1, n$ are easily obtained by the formula in [5, (4.18)]:

$$ \Delta_{w_0\Lambda_{n-1}, s_k\Lambda_{n-1}}(\Theta_{\mathbf{i_0}}^-(c)) = c_{n-1}^{(n-1)}, \qquad \Delta_{w_0\Lambda_n, s_k\Lambda_n}(\Theta_{\mathbf{i_0}}^-(c)) = c_n^{(n-1)}. \tag{9.17} $$

Now, the proof of Theorem 9.1 has been done. $\square$

9.3. **Correspondence to the monomial realizations.** Except for $\Delta_{w_0 s_{n-1}\Lambda_n, \Lambda_{n-1}}(\Theta_{\mathbf{i_0}}^-(c))$ and $\Delta_{w_0 s_n\Lambda_n, \Lambda_n}(\Theta_{\mathbf{i_0}}^-(c))$ (see 9.4), we shall see the positive answer to the conjecture for type D_n.

Let us see the monomial realization of $B(\Lambda_1)$ associated with the cyclic sequence $\cdots(12\cdots n)(12\cdots n)\cdots$, which means that the sign $p_0 = (p_{i,j})$ is given by $p_{i,j} = 1$ if $i < j$ and $p_{i,j} = 0$ if $i > j$ as in the previous sections. The crystal $B(\Lambda_1)$ is described as follows: We abuse the notation $B(\Lambda_1) := \{v_i, v_{\bar{i}} | 1 \le i \le n\}$ if there is no confusion. Then the actions of $\tilde{e}_i$ and $\tilde{f}_i$ $(1 \le i \le n)$ are defined as $\tilde{f}_i = f_i$ and $\tilde{e}_i = e_i$ in (3.11) and (3.12).

To see the monomial realization $B(c_1^{(k)})$, we give the explicit forms of the monomials $A_{i,m}$:

$$ A_{i,m} = \begin{cases} c_1^{(m)} {c_2^{(m)}}^{-1} c_1^{(m+1)} & \text{for } i = 1, \\ c_i^{(m)} {c_{i+1}^{(m)}}^{-1} {c_{i-1}^{(m+1)}}^{-1} c_i^{(m+1)} & \text{for } 1 < i < n-2, \\ c_{n-2}^{(m)} {c_{n-1}^{(m)}}^{-1} {c_n^{(m)}}^{-1} {c_{n-3}^{(m+1)}}^{-1} c_{n-2}^{(m+1)} & \text{for } i = n-2, \\ c_{n-1}^{(m)} {c_{n-2}^{(m+1)}}^{-1} c_{n-1}^{(m+1)} & \text{for } i = n-1, \\ c_n^{(m)} {c_{n-2}^{(m+1)}}^{-1} c_n^{(m+1)} & \text{for } i = n. \end{cases} \tag{9.18} $$

Let where $m_i^{(k)} : B(\Lambda_i) \hookrightarrow \mathcal{Y}(p)$ $(u_{\Lambda_i} \mapsto c_i^{(k)})$ is the embedding of crystal as in Sect.5. Here the monomial realization $B(c_1^{(k)}) = m_1^{(k)}(B(\Lambda_1)) = \{m_1^{(k)}(v_j), m_1^{(k)}(v_{\bar{j}}) | 1 \le j \le n\}$ associated with p_0 is described explicitly:

$$ m_1^{(k)}(v_j) = \begin{cases} \frac{c_j^{(k)}}{c_{j-1}^{(k+1)}} & 1 \le j \le n-2, \\ \frac{c_{n-1}^{(k)}c_n^{(k)}}{c_{n-2}^{(k+1)}} & j = n-1, \\ \frac{c_n^{(k)}}{c_{n-1}^{(k+1)}} & j = n, \end{cases} \qquad m_1^{(k)}(v_{\bar{j}}) = \begin{cases} \frac{c_{j-1}^{(k+n-i)}}{c_j^{(k+n-j)}} & 1 \le j \le n-2, \\ \frac{c_{n-2}^{(k+1)}}{c_{n-1}^{(k+1)}c_n^{(k+1)}} & j = n-1, \\ \frac{c_{n-1}^{(k)}}{c_n^{(k+1)}} & j = n, \end{cases} \tag{9.19} $$

where we understand $c_0^{(k)} = 1$. Now, Theorem 9.1 and Theorem 9.2 claim the following:

Theorem 9.6. *We obtain*

$$\Delta_{w_0\Lambda_k,s_k\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^{n}m_1^{(k)}(v_j)+\sum_{j=k+1}^{n}m_1^{(k)}(v_{\overline{j}}),\qquad (k=1,2\cdots,n-2) \tag{9.20}$$

$$\Delta_{w_0\Lambda_{n-1},s_{n-1}\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c))=m_{n-1}^{(n-1)}(u_{\Lambda_{n-1}}),\qquad \Delta_{w_0\Lambda_n,s_k\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=m_n^{(n-1)}(u_{\Lambda_n}), \tag{9.21}$$

$$\Delta_{w_0s_k\Lambda_k,\Lambda_k}(\Theta^-_{\mathbf{i_0}}(c))=\sum_{j=1}^{k}m_1^{(k-n+1)}(v_{\overline{j}}),\qquad (k=1,2,\cdots,n-2). \tag{9.22}$$

Note that the last result is derived from the fact that $B(c_1^{(k)})=B(c_1^{(n+k-1)^{-1}})$, which is the connected component including $c_1^{(n+k-1)^{-1}}$ as the lowest monomial.

9.4. $\Delta_{w_0s_{n-1}\Lambda_{n-1},\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c))$ **and** $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$**.** To state the results for $\Delta_{w_0s_{n-1}\Lambda_n,\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c))$ and $\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$, we need to prepare the set of triangles $\triangle'_n$ for type D_n which is similar to the one for type B_n:

$$\triangle'_n:=\{(j_k^{(l)}|1\le k\le l<n)|1\le j_k^{(l+1)}\le j_k^{(l)}<j_{k+1}^{(l+1)}\le n\quad(1\le k\le l<n-1)\}. \tag{9.23}$$

We visualize a triangle $(j_k^{(l)})$ in $\triangle'_n$ as follows:

$$(j_k^{(l)})=\qquad\begin{array}{c} j_1^{(1)}\\ j_2^{(2)}\ j_1^{(2)}\\ j_3^{(3)}\ j_2^{(3)}\ j_1^{(3)}\\ \cdots\cdots\cdots\cdots\\ j_{n-1}^{(n-1)}\ \cdots j_2^{(n-1)}\ j_1^{(n-1)}\end{array}$$

Here we know that the set of triangles $\triangle'_n$ for D_n coincides with $\triangle_{n-1}$ for B_n. As type B_n we easily obtain

Lemma 9.7. *For any $k\in\{1,2,\cdots,n-1\}$ there exists a unique j $(1\le j\le k+1)$ such that the kth row of a triangle $(j_k^{(l)})$ in $\triangle'_n$ is in the following form:*

$$k\text{-th row}\quad (j_k^{(k)},j_{k-1}^{(k)},\cdots,j_2^{(k)},j_1^{(k)})=(k+1,k,k-1,\cdots,j+1,j-1,j-2,\cdots,2,1), \tag{9.24}$$

that is, we have $j_m^{(k)}=m$ for $m<j$ and $j_m^{(k)}=m+1$ for $m\ge j$.

For a triangle $\delta=(j_k^{(l)})\in\triangle'_n$, we list j's as in the lemma: $s(\delta):=(s_1,s_2,\cdots,s_{n-1})$, which we call the *label* of a triangle δ. Here we have the following same as Lemma 8.8:

Lemma 9.8. *For any $\delta\in\triangle'_n$ let $s(\delta):=(s_1,s_2,\cdots,s_{n-1})$ be its label. Then*

(*i*) *The label $s(\delta)$ satisfies $1\le s_k\le k+1$ and $s_{k+1}=s_k$ or s_k+1 for $k=1,\cdots,n-1$.*

(*ii*) *Each k-th row of a triangle δ is in one of the following* I, II, III, IV:

I. $s_{k+1}=s_k+1$ *and* $s_k=s_{k-1}$.

II. $s_{k+1}=s_k$ *and* $s_k=s_{k-1}$.

III. $s_{k+1}=s_k+1$ *and* $s_k=s_{k-1}+1$.

IV. $s_{k+1}=s_k$ *and* $s_k=s_{k-1}+1$.

Here we suppose that $s_0=1$ and $s_n=s_{n-2}+1$, which means that the 1st row must be in I,II or IV *and the $n-1$-th row is in* I or IV*.*

Now, we associate a Laurant monomial $m(\delta)$ in variables $(c_i^{(j)})_{i\in I,j\in\mathbb{Z}}$ with a triangle $\delta=(j_k^{(l)})$ by the following way.

(i) Let $s=(s_1\cdots,s_{n-1})$ be the label of $\delta\in\triangle'_n$.

(ii) Suppose i-th row is in the form I. If $1 \le i \le n-2$, then associate $c_i^{(s_i)}$. For $i = n-1$,
   (a) If $n + s_{n-1}$ is even, then associate $c_{n-1}^{(s_{n-1})}$.
   (b) If $n + s_{n-1}$ is odd, then associate $c_n^{(s_{n-1})}$.

(iii) Suppose i-th row is in the form IV. If $1 \le i \le n-2$, then associate $c_i^{(s_i)^{-1}}$. For $i = n-1$,
   (a) If $n + s_{n-1}$ is even, then associate $c_n^{(s_{n-1})^{-1}}$.
   (b) If $n + s_{n-1}$ is odd, then associate $c_{n-1}^{(s_{n-1})^{-1}}$.

(iv) If i-th row is in the form II or III, then associate 1.

(v) Take the product of all monomials as above for $1 \le i < n$, then we obtain the monomial $m(\delta)$ associated with δ. This defines the map $m : \triangle'_n \to \mathcal{Y}$, where $\mathcal{Y}$ is the set of Laurant monomials in $(c_i^{(j)})_{i \in I, j \in \mathbb{Z}}$.

Here we define the involutions ξ and $-$ on $\mathcal{Y}$ by

$$\begin{aligned} &\xi : c_k^{(n-1)} \mapsto c_k^{(n)}, \quad c_k^{(n)} \mapsto c_k^{(n-1)}, c_k^{(j)} \mapsto c_k^{(j)} \quad (j \neq n-1, n), \\ &- : c_i^{(j)} \mapsto c_i^{(n-j)^{-1}}. \end{aligned} \tag{9.25}$$

As type B_n, let us denote the special triangle such that $j_k^{(l)} = k+1$(resp. $j_k^{(l)} = k$) for any k, l by δ_h (reps. δ_l). Indeed, we have

$$m(\delta_h) = \begin{cases} c_{n-1}^{(1)} & \text{if } n \text{ is odd}, \\ c_n^{(1)} & \text{if } n \text{ is even}, \end{cases} \qquad m(\delta_l) = c_n^{(n)^{-1}}. \tag{9.26}$$

Now, we present $\Delta_{w_0 s_{n-1}\Lambda_{n-1},\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c))$ and $\Delta_{w_0 s_n \Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$ for type D_n:

Theorem 9.9. *For type D_n, we have the explicit forms:*

$$\Delta_{w_0 s_n \Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{\delta \in \triangle_n \setminus \{\delta_l\}} \overline{m(\delta)}, \tag{9.27}$$

$$\Delta_{w_0 s_{n-1}\Lambda_{n-1},\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c)) = \sum_{\delta \in \triangle_n \setminus \{\delta_l\}} \overline{\xi \circ m(\delta)}. \tag{9.28}$$

The proof of this theorem will be given in 9.6.

Example 9.10. *The set of triangles $\triangle'_5$ is the same as $\triangle_4$:*

1	1	1	1	2	2	1	1
21	21	21	31	31	31	31	21
321	321	421	421	421	421	421	421
4321	5321	5321	5321	5321	5421	5421	5421
1	2	2	1	2	2	2	2
31	31	32	31	31	32	32	32
431	431	431	431	431	431	432	432
5421	5421	5421	5431	5431	5431	5431	5432

and their labels $s(\delta)$ are

$(2,3,4,5)$, $(2,3,4,4)$, $(2,3,3,4)$, $(2,2,3,4)$, $(1,2,3,4)$, $(1,2,3,3)$, $(2,2,3,3)$, $(2,3,3,3)$,
$(2,2,2,3)$, $(1,2,2,3)$, $(1,1,2,3)$, $(2,2,2,2)$, $(1,2,2,2)$, $(1,1,2,2)$, $(1,1,1,2)$, $(1,1,1,1)$.

Then, we have the corresponding monomials $m(\delta)$:

$$\frac{1}{c_5^{(5)}},\quad \frac{c_5^{(4)}}{c_3^{(4)}},\quad \frac{c_3^{(3)}}{c_2^{(3)}c_4^{(4)}},\quad \frac{c_2^{(2)}}{c_1^{(2)}c_4^{(4)}},\quad \frac{c_1^{(1)}}{c_4^{(4)}},\quad \frac{c_1^{(1)}c_4^{(3)}}{c_3^{(3)}},\quad \frac{c_2^{(2)}c_4^{(3)}}{c_1^{(2)}c_3^{(3)}},\quad \frac{c_4^{(3)}}{c_2^{(3)}},$$
$$\frac{c_3^{(2)}}{c_1^{(2)}c_5^{(3)}},\quad \frac{c_1^{(1)}c_3^{(2)}}{c_2^{(2)}c_5^{(3)}},\quad \frac{c_2^{(1)}}{c_5^{(3)}},\quad \frac{c_5^{(2)}}{c_1^{(2)}},\quad \frac{c_1^{(1)}c_5^{(2)}}{c_2^{(2)}},\quad \frac{c_2^{(1)}c_5^{(2)}}{c_3^{(2)}},\quad \frac{c_3^{(1)}}{c_4^{(2)}},\quad c_4^{(1)}.$$

And then, we have the corresponding monomials $\overline{m(\delta)}$:

$$c_5^{(0)},\quad \frac{c_3^{(1)}}{c_5^{(1)}},\quad \frac{c_2^{(2)}c_4^{(1)}}{c_3^{(2)}},\quad \frac{c_1^{(3)}c_4^{(1)}}{c_2^{(3)}},\quad \frac{c_4^{(1)}}{c_1^{(4)}},\quad \frac{c_3^{(2)}}{c_1^{(4)}c_4^{(2)}},\quad \frac{c_1^{(3)}c_3^{(2)}}{c_2^{(3)}c_4^{(2)}},\quad \frac{c_2^{(2)}}{c_4^{(2)}},$$
$$\frac{c_1^{(3)}c_5^{(2)}}{c_3^{(3)}},\quad \frac{c_2^{(3)}c_5^{(2)}}{c_1^{(4)}c_3^{(3)}},\quad \frac{c_5^{(2)}}{c_2^{(4)}},\quad \frac{c_1^{(3)}}{c_5^{(3)}},\quad \frac{c_2^{(3)}}{c_1^{(4)}c_5^{(3)}},\quad \frac{c_3^{(3)}}{c_2^{(4)}c_5^{(3)}},\quad \frac{c_4^{(3)}}{c_3^{(4)}},\quad \frac{1}{c_4^{(4)}}.$$

Thus, we have

$$\begin{aligned}\Delta_{w_0 s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c)) &= \frac{c_3^{(1)}}{c_5^{(1)}}+\frac{c_2^{(2)}c_4^{(1)}}{c_3^{(2)}}+\frac{c_1^{(3)}c_4^{(1)}}{c_2^{(3)}}+\frac{c_4^{(1)}}{c_1^{(4)}}+\frac{c_3^{(2)}}{c_1^{(4)}c_4^{(2)}}+\frac{c_1^{(3)}c_3^{(2)}}{c_2^{(3)}c_4^{(2)}}+\frac{c_2^{(2)}}{c_4^{(2)}}\\ &+\frac{c_1^{(3)}c_5^{(2)}}{c_3^{(3)}}+\frac{c_2^{(3)}c_5^{(2)}}{c_1^{(4)}c_3^{(3)}}+\frac{c_5^{(2)}}{c_2^{(4)}}+\frac{c_1^{(3)}}{c_5^{(3)}}+\frac{c_2^{(3)}}{c_1^{(4)}c_5^{(3)}}+\frac{c_3^{(3)}}{c_2^{(4)}c_5^{(3)}}+\frac{c_4^{(3)}}{c_3^{(4)}}+\frac{1}{c_4^{(4)}}.\end{aligned}$$

9.5. **Crystal structure on $\triangle'_n$.** We shall define certain crystal structure on $\triangle'_n$ by the similar way to type B_n. First, let us define the actions of $\tilde{f}_i$ and $\tilde{e}_i$ as follows: For a triangle $\delta\in\triangle'_n$ let $J_k=(j^{(k)}_k,j^{(k)}_{k-1},\cdots,j^{(k)}_2,j^{(k)}_1)$ be the k-th row of δ. Thus, we denote $\delta=(J_1,\cdots,J_{n-1})$. It follows from Lemma 9.8 that there exists a unique j such that $J_k=(k+1,k,\cdots,j+1,j-1,\cdots,2,1)$ and J_k is in one of I,II,III,IV. Set $J'_k=(k+1,\cdots,j+2,j,j-1,\cdots,2,1)$ and $J''_k=(k+1,\cdots,j+1,j,j-2,\cdots,2,1)$. Then, we have

$$\tilde{f}_i\delta=\begin{cases}(\cdots,J_{i-1},J'_i,J_{i+1},\cdots) & \text{if } J_i \text{ is in I},\\ 0 & \text{otherwise},\end{cases}\qquad (i=1,\cdots,n-2), \tag{9.29}$$

$$\tilde{f}_{n-1}\delta=\begin{cases}(\cdots,J_{n-2},J'_{n-1}) & \text{if } J_{n-1} \text{ is in I and } n+j \text{ is even},\\ 0 & \text{otherwise},\end{cases} \tag{9.30}$$

$$\tilde{f}_{n}\delta=\begin{cases}(\cdots,J_{n-2},J'_{n-1}) & \text{if } J_{n-1} \text{ is in I and } n+j \text{ is odd},\\ 0 & \text{otherwise},\end{cases} \tag{9.31}$$

$$\tilde{e}_i\delta=\begin{cases}(\cdots,J_{i-1},J''_i,J_{i+1},\cdots) & \text{if } J_i \text{ is in IV},\\ 0 & \text{otherwise},\end{cases}\qquad (i=1,\cdots,n-2), \tag{9.32}$$

$$\tilde{e}_{n-1}\delta=\begin{cases}(\cdots,J_{n-2},J''_{n-1}) & \text{if } J_{n-1} \text{ is in IV and } n+j \text{ is odd},\\ 0 & \text{otherwise},\end{cases} \tag{9.33}$$

$$\tilde{e}_{n}\delta=\begin{cases}(\cdots,J_{n-2},J''_{n-1}) & \text{if } J_{n-1} \text{ is in IV and } n+j \text{ is even},\\ 0 & \text{otherwise}.\end{cases} \tag{9.34}$$

The weight of $\delta=(J_1,\cdots,J_{n-1})$ is defined as follows: Let $s=(s_k)_{k=1,\cdots,n-1}$ be the label of δ, that is, $J_k=(k+1,k,\cdots,j+1,j-1,\cdots,2,1)$ for $j=s_k$.

$$\mathrm{wt}(\delta)=\begin{cases}\Lambda_n-\sum_{k=1}^{n-2}(s_k-1)\alpha_k-\left[\frac{s_{n-1}-1}{2}\right]\alpha_{n-1}-\left[\frac{s_{n-1}}{2}\right]\alpha_n & \text{if } n \text{ is even,}\\ \Lambda_{n-1}-\sum_{k=1}^{n-2}(s_k-1)\alpha_k-\left[\frac{s_{n-1}}{2}\right]\alpha_{n-1}-\left[\frac{s_{n-1}-1}{2}\right]\alpha_n & \text{if } n \text{ is odd,}\end{cases} \tag{9.35}$$

where $[x]$ is the so-called Gaussian symbol, *i.e.*,, the maximum integer which does not exceed x. We can easily check that $\triangle'_n$ is equipped with the crystal structure and obtain.

Proposition 9.11. *We have the following isomorphism of crystals:*

$$\triangle'_n\cong\begin{cases}B(\Lambda_n) & \textit{if } n \textit{ is even,}\\ B(\Lambda_{n-1}) & \textit{if } n \textit{ is odd.}\end{cases}$$

The highest weight crystal is $\delta_h\in\triangle'_n$.

Proof. As was given in 3.4, we know the explicit form of the crystals $B(\Lambda_n)\cong B_{sp}^{(+,n)}$ and $B(\Lambda_{n-1})\cong B_{sp}^{(-,n)}$. So, we shall see how $\triangle'_n$ and $B_{sp}^{(\pm,n)}$ correspond to each other. Let $s=(s_1,\cdots,s_{n-1})$ be the label of a triangle $\delta\in\triangle'_n$. Now, let us associate $\epsilon=(\epsilon_1,\cdots,\epsilon_n)\in B_{sp}^{(\pm,n)}$ with s by

$$\epsilon_1=\begin{cases}+ & \text{if } s_1=1,\\ - & \text{if } s_1=2,\end{cases}\qquad \epsilon_k=\begin{cases}+ & \text{if } s_k=s_{k-1},\\ - & \text{if } s_k=s_{k-1}+1,\end{cases}(2\le k\le n-2), \tag{9.36}$$

$$(\epsilon_{n-1},\epsilon_n)=\begin{cases}(+,+) & \text{if } s_{n-1}=s_{n-2} \text{ and } n+s_{n-1} \text{ is odd.}\\ (+,-) & \text{if } s_{n-1}=s_{n-2} \text{ and } n+s_{n-1} \text{ is even,}\\ (-,+) & \text{if } s_{n-1}=s_{n-2}+1 \text{ and } n+s_{n-1} \text{ is odd,}\\ (-,-) & \text{if } s_{n-1}=s_{n-2}+1 \text{ and } n+s_{n-1} \text{ is even,}\end{cases} \tag{9.37}$$

which define the map $\psi_+:\triangle'_n\to B_{sp}^{(+,n)}$ if n is even and $\psi_-:\triangle'_n\to B_{sp}^{(-,n)}$ if n is odd. Then, e.g., for even n, the vector δ_h (resp. δ_l) corresponds to the highest (resp. lowest) weight vector $(+,+,\cdots,+)$ (resp. $(-,-,\cdots,-)$) since $s(\delta_h)=(1,1,\cdots,1)$ and $s(\delta_l)=(2,3,\cdots,n)$. It is clear to find that the map $\psi_\pm$ is bijective. The rest of the proof is almost the same as the one for Proposition 8.11. □

Next, let us show that the map $m:\triangle'_n\to\mathcal{Y}$ gives an isomorphism between $\triangle'_n$ and the monomial realization of $B(\Lambda_n)$ for even n or $B(\Lambda_{n-1})$ for odd n. Consider the crystal structure on $\mathcal{Y}$ by taking the sign $\overline{p}=(\overline{p}_{i,j})$ which is the same one as (8.30). Then, it corresponds to the cyclic sequence of I such as: $\cdots(nn-1\cdots21)(nn-1\cdots21)\cdots$. Here, by the map m we get

$$m(\delta_h)=\begin{cases}c_n^{(1)} & \text{if } n \text{ is even,}\\ c_{n-1}^{(1)} & \text{if } n \text{ is odd.}\end{cases}$$

Proposition 9.12. *Let $B(c_n^{(1)})$(resp. $B(c_{n-1}^{(1)})$) be the connected subcrystal of $\mathcal{Y}(\overline{p})$ whose highest monomial is $c_n^{(1)}$ (resp. $c_{n-1}^{(1)}$) and which is isomorphic to $B(\Lambda_n)$ (resp.$B(\Lambda_{n-1})$) of type D_n. Then we have $m(\triangle'_n)=B(c_n^{(1)})$ if n is even and $m(\triangle'_n)=B(c_{n-1}^{(1)})$ if n is odd.*

Proof. By the above setting for $\mathcal{Y}(\overline{p})$, we have

$$A_{i,m} = \begin{cases} c_1^{(m)} {c_2^{(m+1)}}^{-1} c_1^{(m+1)}, & \text{for } i = 1, \\ c_i^{(m)} {c_{i-1}^{(m)}}^{-1} {c_{i+1}^{(m+1)}}^{-1} c_i^{(m+1)}, & \text{for } 1 < i < n-2, \\ c_{n-2}^{(m)} {c_{n-3}^{(m)}}^{-1} {c_{n-1}^{(m+1)}}^{-1} {c_n^{(m+1)}}^{-1} c_{n-2}^{(m+1)}, & \text{for } i = n-2, \\ c_{n-1}^{(m)} {c_{n-2}^{(m)}}^{-1} c_{n-1}^{(m+1)}, & \text{for } i = n-1, \\ c_n^{(m)} {c_{n-2}^{(m)}}^{-1} c_n^{(m+1)} & \text{for } i = n. \end{cases} \tag{9.38}$$

For $\delta \in \triangle'_n$ and $i = 1, 2, \cdots, n-2$ we can see that $m(\tilde{f}_i\delta) = \tilde{f}_i(m(\delta))$ by the same way as in the proof of Proposition 8.12.

Thus, let us see $\tilde{f}_{n-1}$ and $\tilde{f}_n$. First, for the label $s(\delta) = (s_1, \cdots, s_{n-1})$, suppose that $n + s_{n-1}$ is odd. Thus, the action of $\tilde{f}_{n-1}$ is trivial and

$$\tilde{f}_n : (s_{n-3}, s_{n-2}, s_{n-1}) = \begin{array}{ccccccc} (j-1, j, j) & \to & (j-1, j, j+1) & & \frac{c_n^{(j)}}{c_{n-2}^{(j)}} & \to & \frac{1}{c_n^{(j+1)}} \\ (j, j, j) & \to & (j, j, j+1) & & c_n^{(j)} & \to & \frac{c_{n-1}^{(j)}}{c_n^{(j+1)}}. \end{array}$$

This means that the action of $\tilde{f}_n$ involves the multiplication of the monomial ${A_{n,j}}^{-1} = {c_n^{(j)}}^{-1} c_{n-2}^{(j)} {c_n^{(j+1)}}^{-1}$. Therefore, we have $m(\tilde{f}_n\delta) = \tilde{f}_n m(\delta)$.

For $s = (s_1, \cdots, s_{n-1})$, suppose that $n + s_{n-1}$ is even. Thus, the action of $\tilde{f}_n$ is trivial and

$$\tilde{f}_{n-1} : (s_{n-3}, s_{n-2}, s_{n-1}) = \begin{array}{ccccccc} (j-1, j, j) & \to & (j-1, j, j+1) & & \frac{c_{n-1}^{(j)}}{c_{n-2}^{(j)}} & \to & \frac{1}{c_{n-1}^{(j+1)}} \\ (j, j, j) & \to & (j, j, j+1) & & c_{n-1}^{(j)} & \to & \frac{c_{n-1}^{(j)}}{c_{n-1}^{(j+1)}}, \end{array}$$

which means that the action of $\tilde{f}_{n-1}$ involves the multiplication of the monomial ${A_{n-1,j}}^{-1} = {c_{n-1}^{(j)}}^{-1} c_{n-2}^{(j)} {c_{n-1}^{(j+1)}}^{-1}$. Therefore, we have $m(\tilde{f}_{n-1}\delta) = \tilde{f}_{n-1} m(\delta)$. As for the actions of $\tilde{e}_i$, we can show that $m(\tilde{e}_i\delta) = \tilde{e}_i m(\delta)$ by the similar way to the case of $\tilde{f}_i$. □

Then, by these results we have

Theorem 9.13. *$B(c_n^{(1)}) = m(\triangle'_n) \cong B(\Lambda_n)$ if n is even and $B(c_{n-1}^{(1)}) = m(\triangle'_n) \cong B(\Lambda_{n-1})$ if n is odd.*

9.6. **Proof of Theorem 9.9.** By the explicit descriptions in (3.14) and (3.15), we know that $e_i^2 = f_i^2 = 0$ on $V_{sp}^{(\pm,n)}$. Thus, we can write $\boldsymbol{x}_i(c) := \alpha_i^\vee(c^{-1})x_i(c) = c^{-h_i}(1 + c \cdot e_i)$ and $\boldsymbol{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) = (1 + c \cdot f_i)c^{-h_i}$ on $V_{sp}^{(\pm,n)}$ and then for $\epsilon = (\epsilon_1, \cdots, \epsilon_n) \in B_{sp}^{(\pm,n)}$

(9.39)

$$\boldsymbol{x}_i(c)\epsilon = \begin{cases} c\epsilon + \epsilon' & \text{if } (\epsilon_i, \epsilon_{i+1}) = (-, +), \\ c^{\frac{\epsilon_{n-1} - \epsilon_n}{2}}\epsilon & \text{otherwise }, \end{cases} \quad (i < n), \qquad \boldsymbol{x}_n(c)\epsilon = \begin{cases} c\epsilon + \epsilon'' & \text{if } (\epsilon_{n-1}, \epsilon_n) = (-, -), \\ c^{-\frac{\epsilon_{n-1} + \epsilon_n}{2}}\epsilon & \text{otherwise }, \end{cases}$$

where $\epsilon' = (\cdots, \epsilon_{i-1}, +, -, \epsilon_{i+2}, \cdots)$ for $i < n$ and $\epsilon'' = (\cdots, \epsilon_{n-2}, +, +)$.

For $c^{(i)} = (c_1^{(i)}, \cdots, c_n^{(i)}) \in (\mathbb{C}^\times)^n$ set $X^{(i)}(c^{(i)}) := \boldsymbol{x}_1(c_1^{(i)}) \cdots \boldsymbol{x}_n(c_n^{(i)})$ and $X(c) := X^{(n-1)}(c^{(n-1)}) \cdots X^{(1)}(c^{(1)})$ where $c := (c^{(n-1)}, \cdots, c^{(1)})$.

To obtain the explicit form of $\Delta_{w_0 s_n \Lambda_n, \Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))$, let us see $\Delta_{w_0 \Lambda_n, s_n \Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c))$ as in 8.7, since for $g \in U\overline{w}_0 U$ we have

$$\Delta_{w_0 s_i \Lambda_i, \Lambda_i}(g) = \Delta_{w_0 \Lambda_i, s_i \Lambda_i}(\eta(g)), \tag{9.40}$$

and $\eta(\Theta^-_{\mathbf{i_0}}(c))=\Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})$ $(-:c^{(i)}_j\mapsto c^{(i)}_{n-j+1})$ as the previous case. Since $\omega(\Theta^-_{\mathbf{i_0}^{-1}}(c))=X(c)$, we have $\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(c))=\langle s_n\epsilon_h,X(c)\epsilon_l\rangle$ where $s_n\epsilon_h=s_n(+,\cdots,+)=(+,\cdots,+,-,-)$ and

$$\epsilon_l=\begin{cases}(-,\cdots,-,-) & \text{if } n \text{ is even,}\\ (-,\cdots,-,+) & \text{if } n \text{ is odd.}\end{cases}$$

Thus, our aim is to obtain the coefficient of the vector $s_n\epsilon_h=(+,\cdots,+,-,-)$ in $X(c)\epsilon_l$.

For the sequence $\mathbf{i_0}=(12\cdots n)^{n-1}$, let J be the set of all subsequences of $\mathbf{i_0}$. The operator $X(c)$ is expanded in the form on $V^{(\pm)}_{sp}$:

$$X(c)=\sum_{\xi\in J}\alpha_\xi e_\xi, \tag{9.41}$$

where $e_\xi=e_{i_1}\cdots e_{i_m}$ for $\xi=(i_1,\cdots,i_m)$ and α_ξ is a product of $\alpha^\vee_i(c)$'s and some scalars. Now, we shall see which sequence $(i_1,\cdots,i_k)\in J$ gives

$$e_{i_1}\cdots e_{i_k}\epsilon_l=(+,\cdots,+,-,-)=s_n\epsilon_h. \tag{9.42}$$

As for the weight, we have

$$\mathrm{wt}(s_n\epsilon_h)-\mathrm{wt}(\epsilon_l)=\sum_{i=1}^{n-2}i\alpha_i+\left[\frac{n-1}{2}\right]\alpha_{n-1}+\left[\frac{n-2}{2}\right]\alpha_n, \tag{9.43}$$

where $[x]$ is the Gaussian symbol as before. Thus, for $(i_1,\cdots,i_k)$ in (9.42), we know that if $i\neq n-1,n$ the number of i's is i and the number of $n-1$ (resp. n) is $\frac{n-2}{2}$ (resp. $\frac{n-2}{2}$) if n is even or $\frac{n-1}{2}$ (resp. $\frac{n-3}{2}$) if n is odd.

Let us associate an element in $U(\mathfrak{n}_+)$ with a triangle $\delta=(j^{(l)}_k)\in\triangle'_n$ by the following way, which is almost same as the one for the type B_n in 8.7:

(i) For $i=1,\cdots,n-1$ define $r_i(\delta)$ to be the set of indices as $r_i(\delta):=\{(k,l)|j^{(l)}_k=i\}$ and $n_i(\delta):=\sharp r_i(\delta)$.
(ii) Set $L=n_i(\delta)$. For $r_i(\delta)$, define $l_{a_1},\cdots,l_{a_L}$ by setting $r_i(\delta)=\{(k_{a_1},l_{a_i}),\cdots,(k_{a_L},l_{a_L})\}$ with $l_{a_1}<l_{a_2}<\cdots<l_{a_L}$.
(iii) For $i=1,2,\cdots,n-1$ set

$$E_i(\delta):=e_{l_{a_1}}e_{l_{a_2}}\cdots e_{l_{a_{L-1}}}\times\begin{cases}e_{l_{a_L}} & \text{if } l_{a_L}\neq n-1,\\ e_{n-1} & \text{if } l_{a_L}=n-1 \text{ and } k_{a_L}+l_{a_L} \text{ is odd,}\\ e_n & \text{if } l_{a_L}=n-1 \text{ and } k_{a_L}+l_{a_L} \text{ is even.}\end{cases}$$

and $E(\delta):=E_{n-1}(\delta)\cdot E_{n-2}(\delta)\cdots E_1(\delta)$.

Example 9.14. *For $n=5$ and a triangle*

$$\delta=\begin{matrix}2\\32\\431\\5421\end{matrix}\qquad \textit{we obtain}\qquad \begin{array}{ll} r_1(\delta)=\{(1,3),(1,4)\}, & E_1(\delta)=e_3e_4,\\ r_2(\delta)=\{(1,1),(1,2),(2,4)\}, & E_2(\delta)=e_1e_2e_5\\ r_3(\delta)=\{(2,2),(2,3)\}, & E_3(\delta)=e_2e_3\\ r_4(\delta)=\{(3,3),(3,4)\}, & E_4(\delta)=e_3e_4,\end{array}$$

and then $E(\delta)=(e_3e_4)(e_2e_3)(e_1e_2e_5)(e_3e_4)$. This example is quite similar to Example 8.14, but $r_2(\delta)$ differs from the one in Example 8.14.

Lemma 9.15. *Let $(j_1,\cdots,j_m)\in J$ be a subsequence of $\mathbf{i_0}$ and $E=e_{j_1}\cdots e_{j_m}$ be the associated monomial of e_i's. Then we have*

$$E\epsilon_l=s_n\epsilon_h=(+,\cdots,+,-,-) \textit{ if and only if there exists } \delta\in\triangle_n\setminus\{\delta_l\} \textit{ such that } E=E(\delta). \tag{9.44}$$

Note that $\delta = (j_k^{(l)}) \neq \delta_l$ if and only if $j_{n-1}^{(n-1)} = n$.

Proof. The proof is similar to the one for Lemma 8.15. Indeed, we can easily see that $E(\delta_l)\epsilon_l \neq s_n\epsilon_h$. Suppose that there exists $\delta \in \triangle'_n \setminus \{\delta_l\}$ such that $E = E(\delta)$. Writing $E(\delta) = e_{l_m} \cdots e_{l_1}$ we may show that $e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$ for any $k = 1, 2, \cdots, m$ by the induction as in the case B_n. For this δ, suppose that e_{l_k} appears in $E_j(\delta)$. The cases $i = l_k = 1, 2, \cdots, n-2$ are done by the same way as the proof of Lemma 8.15. So, let us see the case $i = l_k = n-1$ or n. The $n-2$-th row and the $n-1$-th row of δ around j are as follows:

$$(i)\ \begin{array}{lcc} n-2\text{-th row} & * & j-1 \\ n-1\text{-th row} & j & \quad j-1 \end{array} \qquad (ii)\ \begin{array}{lcc} n-2\text{-th row} & * & j-1 \\ n-1\text{-th row} & j & \quad j-2 \end{array} \qquad (iii)\ \begin{array}{lcc} n-2\text{-th row} & * & j-2 \\ n-1\text{-th row} & j & \quad j-2 \end{array}$$

where $*$ means j or $j+1$ and j is the entry which gives $e_{l_k} = e_{n-1}$ or e_n.

As for (i), we have the following cases:

(a) $E_j(\delta) = \cdots e_n \cdots$ and $E_{j-1}(\delta) = \cdots e_{n-2}e_{n-1} \cdots$.
(b) $E_j(\delta) = \cdots e_{n-1} \cdots$ and $E_{j-1}(\delta) = \cdots e_{n-2}e_n \cdots$.

Now, we consider the case (a). Suppose that $j-1$ in the n-th row gives $e_{n-1} = e_{l_p}$ in $E_{j-1}(\delta)$. Denote the vector before being applied $e_{n-1} = e_{l_p}$ by $\epsilon' = (\epsilon'_i)$, that is, $\epsilon' = e_{l_{p-1}} \cdots e_{l_1}\epsilon_l$, which satisfies $\epsilon'_{n-2} = \epsilon'_{n-1} = -, \epsilon'_n = +$ and then

$$(\epsilon'_{n-2}, \epsilon'_{n-1}, \epsilon'_n) = (-,-,+) \xrightarrow{e_{n-1}} (-,+,-) \xrightarrow{e_{n-2}} (+,-,-).$$

This shows that the vector before being applied $e_n = e_{l_k}$, say $\epsilon'' = (\epsilon''_i)$, has the form $\epsilon''_{n-1} = \epsilon''_n = -$ and then we have $e_n\epsilon'' = e_{l_k} \cdots e_{l_1}\epsilon_l \neq 0$. The case (b) is also shown similarly and the cases (ii) and (iii) are also shown similarly. Therefore, we obtain $E(\delta)\epsilon_l \neq 0$ for any $\delta \in \triangle'_n \setminus \{\delta_l\}$.

Assuming $E\epsilon_l = s_n\epsilon_h = (+, \cdots, +, -, -)$ we see that each e_i $(i \neq n)$ should appear in E i-times $(1 \leq i \leq n-2)$, e_{n-1} should appear $\left[\frac{n-1}{2}\right]$-times and e_n should appear $\left[\frac{n-2}{2}\right]$-times by (9.43). Moreover, e_{n-1} and e_n appear alternatively since only e_n can change $\epsilon_n = -$ to $+$ and only e_{n-1} can change $\epsilon_n = +$ to $-$. In the sequence $\mathbf{i_0}$ there are $n-1$-cycles $12 \cdots n$ just as

$$\mathbf{i_0} = \underbrace{12\cdots n}_{(n-1)\text{th cycle}} \quad \cdots \quad \underbrace{12\cdots n}_{2\text{nd cycle}} \quad \underbrace{12\cdots n}_{1\text{st cycle}}$$

Since the indices $n-1$ and n appear alternatively $\left[\frac{n-1}{2}\right] + \left[\frac{n-2}{2}\right] = n-2$-times in $\mathbf{i_0}$, we should choose $n-1$ or n from one cycle alternatively. Listing the number of such cycles from the right, we obtain $(n-1, \cdots, j+1, j-1, \cdots, 2, 1) = (j_{n-2}^{(n-1)}, j_{n-3}^{(n-1)}, \cdots, j_2^{(n-1)}, j_1^{(n-1)})$ for some j, which is the bottom row($= n-1$-th row) of an element in $\triangle'_n$ except the first entry $j_{n-1}^{(n-1)} = n$. Next, we define the $n-2$-th row. We can apply e_{n-2} after applying e_{n-1} or e_n and can apply e_{n-1} or e_n after applying e_{n-2}. Thus, this means that $n-2$ must be in between $n-1$ and n chosen in the above process. Then, we obtain the $n-2$-th row $(j_{n-2}^{(n-2)}, j_{n-3}^{(n-2)}, \cdots, j_2^{(n-2)}, j_1^{(n-2)})$, which should satisfy the condition $j_k^{(n-1)} \leq j_k^{(n-2)} < j_{k+1}^{(n-1)}$ $(1 \leq k \leq n-2)$ since this $n-2$ can be in the same cycle as the previous $n-1$ or n. This condition just coincides with the ones for $\triangle'_n$. Repeating this process, we obtain a triangle δ and find that this $\delta \in \triangle'_n$ satisfies $E = E(\delta)$. □

Next, let us find the coefficient of the vector generated from ϵ_l by applying $a_\xi e_{i_1} \cdots e_{i_k}$ in (8.34). Set $\mathbf{E}_i(c) = c\alpha_i^\vee(c^{-1})e_i$. Then we can write $\boldsymbol{x}_i(c) = \alpha_i^\vee(c^{-1}) + \mathbf{E}_i(c)$. We consider the cases that δ is in I$\sim$IV. In the case that δ is in I, there exists j such that the $i-1$, i, $i+1$th rows of δ are in the form:

$$\begin{array}{lllllll} i-1\text{-th row} & \cdots j+3 & j+2 & j+1 & j-1 & j-2 & \\ i\text{-th row} & \cdots & j+3 & j+2 & j+1 & j-1 & j-2 \\ i+1\text{-th row} & \cdots & \quad j+3 & j+2 & j & j-1 & j-2 \end{array}$$

This means that in the expansion (9.41) we know that this δ gives the following monomial in which $\alpha_i^\vee$ appears once:

$$\cdots\alpha_i^\vee({c_i^{(j)}}^{-1})\mathbf{E}_{i+1}(c_{i+1}^{(j)})\cdots\mathbf{E}_{i-1}(c_{i-1}^{(j-1)})\mathbf{E}_i(c_i^{(j-1)})\mathbf{E}_{i+1}(c_{i+1}^{(j-1)})\cdots \tag{9.45}$$

Hence, by the action of $\alpha_i^\vee({c_i^{(j)}}^{-1})$ we obtain the coefficient $c_i^{(j)}$. In the other cases II,III,IV, we can discuss similarly and obtain that for II and III, we have the coefficient 1 and for IV we have ${c_i^{(j)}}^{-1}$, which coincides with the recipe after Lemma 9.8 and then we know that the desired coefficient is the same as $m(\delta)$.

As has been seen in Proposition 9.11, the set of monomials $m(\triangle'_n)$ has the crystal structure isomorphic to $B(\Lambda_{n-1})$ or $B(\Lambda_n)$. The following lemma is immediate from Theorem 5.4.

Lemma 9.16. *The set of monomials $\overline{m(\triangle'_n)}$ (resp. $\overline{\xi\circ m(\triangle'_n)}$) is a crystal isomorphic to $B(c_n^{(0)})\cong B(\Lambda_n)$ (resp. $B(\Lambda_{n-1})$), where $^-$ and ξ are the involutions as in* (9.25).

Proof. We find by the definition of the map $^-$ that this is the case $a=n$ in Theorem 5.4. By Proposition 9.11, we know that $\triangle'_n\cong B(c_n^{(1)})$ if n is even and $\triangle'_n\cong B(c_{n-1}^{(1)})$ if n is odd. Thus, by Theorem 5.4 we have that for an even n, $\overline{B(c_n^{(1)})}=B({c_n^{(n-1)}}^{-1})=B(c_n^{(0)})$ and for an odd n, $\overline{B(c_{n-1}^{(1)})}=B({c_{n-1}^{(n-1)}}^{-1})=B(c_n^{(0)})$, which means the desired result. □

Thus, by the formula (9.40), we obtain

$$\Delta_{w_0s_n\Lambda_n,\Lambda_n}(\Theta^-_{\mathbf{i_0}}(c))=\Delta_{w_0\Lambda_n,s_n\Lambda_n}(\Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})),$$

where for $c=(c^{(1)},\cdots,c^{(n)})$ ($c^{(i)}=(c_l^{(i)})_{1\le l\le n}$) we define $\overline{c}=(\overline{c^{(n-1)}},\cdots,\overline{c^{(1)}})$ ($\overline{c^{(i)}}=({c_{n-l+1}^{(i)}}^{-1})_{1\le l\le n}$). The case $k=n-1$

$$\Delta_{w_0s_{n-1}\Lambda_{n-1},\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}}(c))=\Delta_{w_0\Lambda_{n-1},s_{n-1}\Lambda_{n-1}}(\Theta^-_{\mathbf{i_0}^{-1}}(\overline{c})),$$

is also obtained by considering the map ξ, that is, by flipping $n-1\leftrightarrow n$. Therefore, we have completed the proof of Theorem 9.9. □

Hence, we get the positive answer to our conjecture for type D_n.

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Department of Mathematics, Sophia University, Kioicho 7-1, Chiyoda-ku, Tokyo 102-8554, Japan
E-mail address: toshiki@sophia.ac.jp